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Analysis III
8. Exercise: Vector Fields

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Preparation Exercises

47. Coordinate vector fields.

Let $\phi : U \to \mathbb{R}^n$ be a chart of X for an open set $U \subset X$. Then consider the vector field $F_i : U \to TU$ with

$$F_i(x) = T_x(\phi)^{-1}(e_i) \in T_xU \ ,$$

for $i \in \{1, \ldots, n\}$ and where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ is the i-th standard unit vector of $\mathbb{R}^n$. This is called a coordinate vector field.

(a) Show that these are vector fields $F_i \in \mathrm{Vec}^\infty(U)$.

(b) Show that any other vector field F on U can be written

$$F(x) = \sum_i a_i(x) F_i(x)$$

for smooth functions $a_i : U \to \mathbb{R}$.

Solution.

(a) A vector field is another name for a section of the tangent bundle. The formula given defines a map with $\pi(F_i(x)) = x$. We only have to check that these a smooth. But using the chart $T(\phi)$ for the tangent bundle gives us

$$T(\phi) \circ F_i \circ \phi^{-1} = T_{\phi^{-1}}(\phi) \circ T_{\phi^{-1}(x)}(\phi)^{-1}(e_i) = e_i.$$

This is constant, so indeed smooth.

(b) If we write any other vector field in these coordinates we get

$$T(\phi) \circ F \circ \phi^{-1} : \phi[U] \to \mathbb{R}^n.$$

Since every point in $\mathbb{R}^n$ is a linear combination of the basis vectors we get

$$T(\phi) \circ F \circ \phi^{-1}(y) = \sum_i \tilde{a}(y) e_i.$$

Rearranging

$$F(x) = T(\phi)^{-1} \left(\sum_i \tilde{a}(\phi(x)) e_i \right) = \sum_i \tilde{a}(\phi(x)) T(\phi)^{-1}(e_i),$$

since the tangent map is linear.

48. Vector fields and derivations.

(a) For a vector field F on X, describe the difference and relationship between the derivation θ_F defined by Theorem 2.2 and D_v described by Theorem 1.40.

(b) What is the derivation that corresponds to a coordinate vector field?

(c) Suppose that $F = \sum_i a_i(x)F_i(x)$ as in the previous exercise. Show that $\Theta_F = \sum_i a_i \Theta_{F_i}$.

Solution.

(a) Let $f : X \to \mathbb{R}$ be a function. Just before Theorem 1.40 in the script, for every vector $v \in T_x X$ we define a derivation $D_v : C^\infty(X, \mathbb{R}) \to \mathbb{R}$ at the point x. In essence, given a function and a vector at a point, we get a single number. If we have a vector at every point of X then we get a number at every point of X, ie a function $X \to \mathbb{R}$. This is the definition of the derivation $\theta_F(f)$:

$$\theta_F(f) = x \mapsto D_{F(x)}(f).$$

Conversely, if we have a $\theta_F(f)$ then we can produce a derivation at any point $x \in X$:

$$D(f) = \theta_F(f)(x).$$

(b) To answer this, we need to recall several constructions. Choose any point $x \in X$ and smooth function $f : X \to \mathbb{R}$. Suppose that the coordinate vector field is coming from the chart ϕ. Then we should compute $\theta_{F_i}(f)(x) = D_{F_i(x)}(f)$. The derivation D was defined in terms of curves. $F_i(x)$ is a vector and is represented by the curve $\alpha(t) = \phi^{-1}(te_i + \phi(x))$. So

$$\begin{aligned} D_{F_i(x)}(f) &= J_0(f \circ \alpha) = J_0(f \circ \phi^{-1}(te_i + \phi(x))) \\ &= J_{\phi(x)}(f \circ \phi^{-1})\ J_0(te_i + \phi(x)) \\ &= J_{\phi(x)}(f \circ \phi^{-1})\ e_i \\ &= \frac{\partial (f \circ \phi^{-1})}{\partial y_i}. \end{aligned}$$

Thus we see that the derivation corresponding to a coordinate vector field is partial differentiation.

(c) This can be proved in several ways. The most direct way is that there is a vector space isomorphism between tangent vectors and derivations at a point. Another approach is to notice $\Theta_{F_i}(\phi_j) = 1$ if $i = j$ and 0 otherwise.

The previous exercises all together show that all derivations are (locally) a sum of partial derivatives.

In Class Exercises

49. The Lie bracket in $\mathbb{R}^n$.

The Lie bracket is the name of the operation on vector fields defined in Corollary 2.3. Let us focus on $\mathbb{R}^n$. The tangent bundle is trivial $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ and we can write a vector field as $F : \mathbb{R}^n \to \mathbb{R}^n$ (technically we should write $F(x) = (\tilde{F}(x), x)$, but the tildes are annoying).

(a) How can we calculate $\theta_F(f)$ for some function $f : \mathbb{R}^n \to \mathbb{R}$?

(b) Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields on $\mathbb{R}^n$. Show

$$[F, G](x) = JG(x)\, F(x) - JF(x)\, G(x) \ .$$

(c) Consider the three vector fields on $\mathbb{R}^4$ (we have seen these already in the exercise about $T\mathbb{S}^3$):

$$\begin{aligned} F(x_1, x_2, x_3, x_4) &:= (-x_2, x_1, x_4, -x_3) \ , \\ G(x_1, x_2, x_3, x_4) &:= (-x_3, -x_4, x_1, x_2) \\ \text{and} \quad H(x_1, x_2, x_3, x_4) &:= (-x_4, x_3, -x_2, x_1) \ . \end{aligned}$$

(i) Calculate $[F, G]$, $[G, H]$ und $[F, H]$.

(ii) For these three fields, check that the *Jacobi identity* holds:

$$[F, [G, H]] = [[F, G], H] + [G, [F, H]] \ .$$

Solution.

(a) If we have a tangent vector $v \in \mathbb{R}^n$ at a point $x \in R^n$, then $D_v(f)$ is the directional derivative:

$$D_v(f) = \left.\frac{d}{dt}\right|_{t=0} f(x + vt) = \nabla f \cdot v,$$

because a path representing the tangent vector v is $y(t) = x + vt$. It follows then that $\theta_F(f)(x) = \nabla f(x) \cdot F(x)$.

(b) We firstly calculate $\theta_F \circ \theta_G - \theta_G \circ \theta_F$ and then try to see which vector field it could come from. Let $f : \mathbb{R}^n \to \mathbb{R}$ be any function. Note that

$$\frac{\partial}{\partial x_i}\Big(\nabla f \cdot G\Big) = \frac{\partial}{\partial x_i}\sum_j (\partial_j f)\, G_j = \sum_j (\partial_j \partial_i f)\, G_j + (\partial_j f)\,(\partial_i G_j)$$

Now we can compute half the expression.

$$\begin{aligned}\theta_F \circ \theta_G(f) &= \theta_F\Big(\nabla f \cdot G\Big) = \nabla\Big(\nabla f \cdot G\Big) \cdot F \\ &= \sum_{i,j} (\partial_j \partial_i f)\, G_j\, F_i + (\partial_j f)\,(\partial_i G_j)\, F_i\end{aligned}$$

Swapping F and G gives an expression for $\theta_G \circ \theta_F(f)$ too. The difference is

$$\begin{aligned}\theta_F \circ \theta_G(f) - \theta_G \circ \theta_F(f) &= \sum_{i,j} (\partial_j f)\,(\partial_i G_j)\, F_i - (\partial_j f)\,(\partial_i F_j)\, G_i \\ &= \sum_j (\partial_j f) \sum_i (\partial_i G_j)\, F_i - (\partial_i F_j)\, G_i \\ &= \nabla f \cdot \left(\sum_i (\partial_i G_j)\, F_i - (\partial_i F_j)\, G_i\right)_j \\ &= \nabla f \cdot \Big(\nabla G_j \cdot F - \nabla F_j \cdot G\Big)_j\end{aligned}$$

Thus we see that $[F, G]$ is the vector field whose j-th component has the formula in the bracket. But the Jacobian matrix JF of a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix whose j-th row is the gradient of F_j. Thus this formula is the same as the formula in the question.

(c) We can use the formula we just derived:

$$\begin{aligned}[F, G] &= JG \cdot F - JF \cdot G(x) \\ &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \\ x_4 \\ -x_3 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -x_3 \\ -x_4 \\ x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} -x_4 \\ x_3 \\ -x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} -2x_4 \\ 2x_3 \\ -2x_2 \\ 2x_1 \end{pmatrix}\end{aligned}$$

Similarly we have $[G, H](x) = (-2x_2, 2x_1, 2x_4, -2x_3)$ and $[F, H](x) = (2x_3, 2x_4, -2x_1, -2x_2)$.

(d) For this part, we could go ahead and calculate another three Lie brackets. But notice that in fact $[F, G] = 2H$, $[G, H] = 2F$ and $[F, H] = -2G$. It follows that

$$\begin{aligned}&[F, [G, H]] = [F, 2F] = 2F' \cdot F - F' \cdot 2F = 0, \\ &[[F, G], H] + [G, [F, H]] = [2H, H] + [G, -2G] = 0.\end{aligned}$$

If $[F,G] = 2H$ reminds you of the cross-product in $\mathbb{R}^3$, there's a good reason. Consider these vector fields at the point $x = (1,0,0,0)$. Then

$$\begin{aligned} F(1,0,0,0) &:= (0,1,0,0)\ , \\ G(1,0,0,0) &:= (0,0,1,0) \\ \text{and} \quad H(1,0,0,0) &:= (0,0,0,1)\ . \end{aligned}$$

so we can see these vectors as the basis of $\mathbb{R}^3$ and the Lie bracket as twice the cross-product.

50. The computation of the Lie Bracket for submanifolds of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and $F, G \in \mathrm{Vec}^\infty(X)$. With the help of Theorem 2.22(iii),(iv) devise a formula to compute $[F,G]$. Prove your formula.

Solution. Using Theorem 2.22(iii), extend F and G to vector fields on $\mathbb{R}^n$ called $\tilde{F}, \tilde{G}$. Then by Theorem 2.22(iii) and the exercise on the Lie bracket in $\mathbb{R}^n$ we have that

$$[F,G]_X = [\tilde{F},\tilde{G}]_{\mathbb{R}^n} = J\tilde{G}\cdot\tilde{F} - J\tilde{F}\cdot\tilde{G}.$$

So here we already have a formula that avoids using coordinate charts. There is the practical question of how to find extensions of vector fields on X. If the manifold is a submanifold, many times the formula for the vector field will already be the restriction of formulas on $\mathbb{R}^n$.

If you are in the situation where there is not an easy extension, here is a practical way to construct one. Choose a point $x \in X$. Because X is a submanifold, we know that locally X is the graph of a function $h : U \to \mathbb{R}^{n-k}$. For simplicity, assume it is a graph over the coordinates $y = (x_1,\ldots,x_k)$. In other words, $y \mapsto (y, h(y))$ is the inverse of the chart $\phi(x) = (x_1,\ldots,x_k)$ of X. Thus we can write the vector fields $F(y,h(y)), G(y,h(y))$ in this neighbourhood as functions of y alone. Then $\tilde{F}(x) := F(y,h(y))$ is an extension of F to $U \times \mathbb{R}^{n-k}$, and likewise for $\tilde{G}$. The advantage of this choice of extension is that they are constant in the variables $x_{k+1},\ldots,x_n$, so for example

$$J\tilde{G}\cdot\tilde{F} = \begin{pmatrix} \frac{\partial\tilde{G}_1}{\partial x_1} & \cdots & \frac{\partial\tilde{G}_1}{\partial x_k} & 0\ldots0 \\ \vdots & & \vdots & \\ \frac{\partial\tilde{G}_n}{\partial x_1} & \cdots & \frac{\partial\tilde{G}_n}{\partial x_k} & 0\ldots0 \end{pmatrix} \begin{pmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial\tilde{G}_1}{\partial x_1} & \cdots & \frac{\partial\tilde{G}_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial\tilde{G}_n}{\partial x_1} & \cdots & \frac{\partial\tilde{G}_n}{\partial x_k} \end{pmatrix} \begin{pmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_k \end{pmatrix}$$

$$(\tilde{G}'\tilde{F})_j(x) = \sum_{i=0}^{k} \frac{\partial\tilde{G}_j}{\partial x_i} F_i(x) = \sum_{i=0}^{k} \left(\frac{\partial G_j}{\partial x_i} + \sum_{l=1}^{n-k} \frac{\partial G_j}{\partial x_{k+l}} \frac{\partial h_l}{\partial x_i} \right) F_i(x)$$

The derivatives of h can also be found relatively easily by solving the linear system

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial(x_{k+1}, \dots, x_n)} Jh = 0,$$

where $f(y, h(y)) = c$ describes X in this neighbourhood as a level set. Hence we can compute the Lie bracket at this point x using just the vector fields F, G defined on X and a level set describing X locally. If your submanifold is not defined using level sets, well then it probably has nice charts and you should probably just compute the Lie bracket using them.

Additional Exercises

51. Properties of the Lie bracket. Let X be an n-dimensional manifold.

(a) Show: the Lie bracket has the following properties for all vector fields $F, G, H \in \mathrm{Vec}^\infty(X)$ and scalars $a \in \mathbb{R}$.

(i) $\mathbb{R}$-linear: $[aF, G] = a[F, G]$.

(ii) anti-symmetric: $[F, G] = -[G, F]$.

(iii) Jacobi identity: $[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0$.

Hint: The pairing $F \to \theta_F$ is injective (and for smooth vector fields and derivations it is bijective), so it is enough to show equality for the corresponding derivations. Eg. to show $[aF, G] = a[F, G]$ you can show $\theta_{[aF,G]} = \theta_{a[F,G]}$.

(b) Show that coordinate vector fields commute: $[F_i, F_j] = 0$ for every i, j.

Solution.

(a) The Lie bracket is a local construct, so choose any point $x \in X$ and a chart $\phi : U \to \mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be any smooth function. Applying the definitions of Theorems 1.40 and 2.2 to a general manifold gives

$$\begin{aligned} \theta_F(f) : x \mapsto D_{F(x)}(f) &= \left.\frac{d}{dt}\right|_{t=0} f(\alpha(t)) \text{ for } [\alpha] = F(x) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\phi^{-1}(\phi(x) + vt)) \text{ for } v = T_x(\phi)(F(x)). \end{aligned}$$

The formulas are equivalent, but depending on how the vectors of the vector field are described, whether as curves or in local coordinates, one formula might be easier than the other. Now the derivation $\theta_{[F,G]} = \theta_F \circ \theta_G - \theta_G \circ \theta_F$.

$\mathbb{R}$-linear: Notice that θ_F is $\mathbb{R}$-linear in F:

$$\begin{aligned}\theta_{aF}(f)(x) &= \frac{d}{dt}\bigg|_{t=0} f(\phi^{-1}(\phi(x)+avt)) \text{ for } v = T_x(\phi)(F(x)) \\ &= \frac{d}{d(s/a)}\bigg|_{s=0} f(\phi^{-1}(\phi(x)+vs)) \text{ for } s = at \\ &= a\,\theta_F(f)(x).\end{aligned}$$

And linearity in f follows from the Leibniz rule. Thus

$$\begin{aligned}\theta_{[aF,G]}(f) &= \theta_{aF}\Big(\theta_G(f)\Big) - \theta_G\Big(\theta_{aF}(f)\Big) \\ &= a\,\theta_F\Big(\theta_G(f)\Big) - \theta_G\Big(a\,\theta_F(f)\Big) \\ &= a\,\theta_F\Big(\theta_G(f)\Big) - a\,\theta_G\Big(\theta_F(f)\Big) \\ &= a\,\theta_{[F,G]}(f) = \theta_{a[F,G]}(f)\end{aligned}$$

Anti-symmetry also follows from the linearity of θ_F in F:

$$\theta_{-[G,F]} = -\theta_{[G,F]} = -\theta_G \circ \theta_F + \theta_F \circ \theta_G = \theta_{[F,G]}$$

Finally we must show the Jacobi identity.

$$\begin{aligned}\theta_{[F,[G,H]]}(f) &= \theta_F\Big(\theta_G\theta_H(f) - \theta_H\theta_G(f)\Big) - (\theta_G\theta_H - \theta_H\theta_G)\Big(\theta_F(f)\Big) \\ &= \theta_F\theta_G\theta_H(f) - \theta_F\theta_H\theta_G(f) - \theta_G\theta_H\theta_F(f) + \theta_H\theta_G\theta_F(f).\end{aligned}$$

If you permute the F,G, and H to compute the other two terms, you see that every permutation of $\theta_F\theta_G\theta_H$ occurs twice, once with each sign. Therefore the sum is zero.

(b) Here the second version of the formula for θ_F is very useful, because $v = T_x(\phi)(F_i(x)) = T_x(\phi)T_x(\phi)^{-1}e_i = e_i$ for every point $x \in U$. Then

$$\begin{aligned}\theta_{F_i}\theta_{F_j}(f)(x) &= \theta_{F_i}\left(y \mapsto \frac{d}{dt}\bigg|_{t=0} f(\phi^{-1}(\phi(y)+e_jt))\right)(x) \\ &= \frac{d}{ds}\bigg|_{s=0}\frac{d}{dt}\bigg|_{t=0} f(\phi^{-1}(\phi(x)+e_is+e_jt)) \\ &= \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} f(\phi^{-1}(\phi(x)+e_is+e_jt)) \\ &= \theta_{F_j}\theta_{F_i}(f).\end{aligned}$$

This shows $\theta_{[F_i,F_j]} = 0$, and hence $[F_i, F_j] = 0$.

More explanation/another example: Perhaps it is useful to see how special this property is by doing the same computation for F and G from Exercise 47(c) considered as vector fields on $\mathbb{S}^3$. Choose the point $x_0 = (1, 0, 0, 0)$ and a small neighbourhood $U \subset \mathbb{S}^3$ of this point. Then we can use the chart $\phi(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4)$ which has inverse $\phi^{-1}(y_1, y_2, y_3) = (h(y), y_1, y_2, y_3)$ for $h(y) = +\sqrt{1 - \|y\|^2}$. Let $y = \phi(x)$ be the corresponding local coordinate for any point x. First we compute the vectors in local coordinates

$$\begin{aligned} v_F(y) &:= T_x(\phi)F(x(y)) = (x_1, x_4, -x_3) = (h(y), y_3, -y_2), \\ v_G(y) &:= T_x(\phi)G(x(y)) = (-x_4, x_1, x_2) = (-y_3, h(y), y_1). \end{aligned}$$

Take any smooth function $f : \mathbb{S}^3 \to \mathbb{R}$. The application of θ_F to f is standard:

$$\theta_F(f)(x) = \frac{d}{dt}\bigg|_{t=0} f \circ \phi^{-1}(y + v_F(y)t).$$

Here is the important point. When we apply θ_G to this, we make the substitution $y + v_G(y)s$ for y, but the vector v_F also depends on y! This gives

$$\theta_G\theta_F(f)(x) = \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} f \circ \phi^{-1}\Big(y + v_G(y)s + v_F(y + v_G(y)s)t\Big).$$

Now I think you can see why the order of θ_F and θ_G is important. Let's complete this calculation now:

$$\begin{aligned} y + v_G(y)s &= (y_1 - y_3 s, y_2 + h(y)s, y_3 + y_1 s) \\ v_F(y + v_G(y)s) &= v_F(y_1 - y_3 s, y_2 + h(y)s, y_3 + y_1 s) \\ &= (h(y + v_G(y)s), y_3 + y_1 s, -y_2 - h(y)s)) \\ \theta_G\theta_F(f)(x) &= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} f \circ \phi^{-1}\Big(y + v_G(y)s + v_F(y + v_G(y)s)t\Big) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} f \circ \phi^{-1} \begin{pmatrix} y_1 - y_3 s + h(y + v_G(y)s)t \\ y_2 + h(y)s + (y_3 + y_1 s)t \\ y_3 + y_1 s + (-y_2 - h(y)s)t \end{pmatrix} \end{aligned}$$

Let's assume that f is given as the restriction of a smooth function on $\mathbb{R}^4$, which is always possible, so that we can use vector calculus for the next steps. You can also do this with the chain rule for manifolds with the tangent map instead of the gradient and Jacobian, and of course it is basically the same thing, but I think it is a little clearer to write it this way. We continue

$$\begin{aligned}
\theta_G\theta_F(f)(x) &= \nabla f \cdot J(\phi^{-1}) \cdot \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} y_1 - y_3 s + h(y + v_G(y)s)t \\ y_2 + h(y)s + (y_3 + y_1 s)t \\ y_3 + y_1 s + (-y_2 - h(y)s)t \end{pmatrix} \\
&= \nabla f \cdot J(\phi^{-1}) \cdot \frac{d}{ds}\bigg|_{s=0} \begin{pmatrix} h(y + v_G(y)s) \\ y_3 + y_1 s \\ -y_2 - h(y)s \end{pmatrix} \\
&= \nabla f \cdot J(\phi^{-1}) \cdot \begin{pmatrix} h'(y) \cdot v_G(y) \\ y_1 \\ -h(y) \end{pmatrix}.
\end{aligned}$$

In the same way

$$\begin{aligned}
y + v_F(y)t &= (y_1 + h(y)t, y_2 + y_3 t, y_3 - y_2 t) \\
v_G(y + v_F(y)t) &= (-y_3 + y_2 t, h(y + v_F(y)t), y_1 + h(y)t) \\
\theta_F\theta_G(f)(x) &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f \circ \phi^{-1}\Big(y + v_F(y)t + v_G(y + v_F(y)t)s\Big) \\
&= \nabla f \cdot J(\phi^{-1}) \cdot \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \begin{pmatrix} y_1 + h(y)t + (-y_3 + y_2 t)s \\ y_2 + y_3 t + h(y + v_F(y)t)s \\ y_3 - y_2 t + (y_1 + h(y)t)s \end{pmatrix} \\
&= \nabla f \cdot J(\phi^{-1}) \cdot \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} -y_3 + y_2 t \\ h(y + v_F(y)t) \\ y_1 + h(y)t \end{pmatrix} \\
&= \nabla f \cdot J(\phi^{-1}) \cdot \begin{pmatrix} y_2 \\ h'(y) \cdot v_F(y) \\ h(y) \end{pmatrix}
\end{aligned}$$

Finally we can say

$$\begin{aligned}
\theta_{[F,G]}(f)(x) &= \nabla f \cdot J(\phi^{-1}) \cdot \begin{pmatrix} y_2 - h'(y) \cdot v_G(y) \\ h'(y) \cdot v_F(y) - y_1 \\ 2h(y) \end{pmatrix} \\
&= \nabla f \cdot J(\phi^{-1}) \cdot \begin{pmatrix} x_3 + x_1^{-1}(x_2, x_3, x_4) \cdot (-x_4, x_1, x_2) \\ -x_1^{-1}(x_2, x_3, x_4) \cdot (x_1, x_4, -x_3) - x_2 \\ 2x_1 \end{pmatrix} \\
&= \nabla f \cdot J(\phi^{-1}) \cdot x_1^{-1} \begin{pmatrix} x_1 x_3 - x_2 x_4 + x_1 x_3 + x_2 x_4 \\ -(x_1 x_2 + x_3 x_4 - x_3 x_4) - x_1 x_2 \\ 2x_1^2 \end{pmatrix} \\
&= \nabla f \cdot \begin{pmatrix} -x_1^{-1} x_2 & -x_1^{-1} x_3 & -x_1^{-1} x_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2x_3 \\ -2x_2 \\ 2x_1 \end{pmatrix} = \nabla f \cdot \begin{pmatrix} -2x_4 \\ 2x_3 \\ -2x_2 \\ 2x_1 \end{pmatrix}
\end{aligned}$$

and this is the same answer we found previously.