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Analysis III
7. Exercise: Vector Bundles II

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Preparation Exercises

41. A non-trivial line bundle over $\mathbb{R}/\mathbb{Z}$.

Consider the lines $L_x = \mathbb{R}(\cos \pi x, \sin \pi x) \subset \mathbb{R}^2$. Notice that $L_{x+1} = L_x$.

(a) By writing $M := \{(v, [x]) \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \mid v \in L_x\}$ locally as the level set of functions $F : \mathbb{R}^2 \times U_x \to \mathbb{R}$, prove it is a submanifold.

Hint. How can you write the line L_x as the zero set of a function?

(b) Show M is a vector bundle over $\mathbb{R}/\mathbb{Z}$. It is called the *Möbius band* or *Möbius bundle*.

Hint. Can you find non-vanishing local sections?

(c) Compute the cocycle $g_{U_{0.5}, U_0}$.

(d) Prove that $(M, \mathbb{R}/\mathbb{Z}, \pi)$ is a non-trivial bundle.

Hint. Prove there are no non-vanishing global sections.

Solution.

(a) $\mathbb{R}^2 \times U_x$ is an open set of $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ so it is enough to check that M is a submanifold on these sets (submanifold is a local property).

Following the hint, the line $L_x \subset \mathbb{R}^2$ is the solution set $(-\sin \pi x)v_1 + (\cos \pi x)v_2 = 0$. Therefore take $F(v, [x]) = (-\sin \pi x)v_1 + (\cos \pi x)v_2$. The gradient of this is

$$\nabla F = \Big((-\pi \cos \pi x)v_1 + (-\pi \sin \pi x)v_2, -\sin \pi x, \cos \pi x \Big).$$

This is never zero. By the implicit function theorem/constant rank theorem, $F^{-1}[\{0\}] = M \cap (\mathbb{R}^2 \times U_x)$ is a submanifold.

(b) The projection map is clearly going to be $\pi(v, [x]) = [x]$.

Let $\ell(x) = (\cos \pi x, \sin \pi x) \in L(x)$. Over $U_x \subset \mathbb{R}/\mathbb{Z}$ we have the local section $s_x = \ell \circ \phi_x$. This is smooth because in coordinates it is

$$(\Pi_i \times \pi) \circ s_x \circ \phi_x^{-1} = (\Pi_i \times \pi) \circ \ell$$

It is also never 0.

Non-vanishing local sections are basically equivalent to local trivialisations. $\Phi_x(t, [y]) = ts_x(y)$ which maps $\mathbb{R} \times U_x \to \pi^{-1}[U_x]$.

(c) Another way check something is a vector bundle is in terms of cocycles. $U_0 \cup U_{0.5} = \mathbb{R}/\mathbb{Z}$ so it sufficient to give $g_{U_{0.5}, U_0}$ to determine the bundle, since all the cocycle conditions are automatically fulfilled if there is only one element in the cocycle.

We need to calculate $\Phi_{0.5}^{-1} \circ \Phi_0$. For $x \in (0, 0.5)$

$$\Phi_0(t, [x]) = t\ell(\phi_0([x])) = t\ell(x)$$
$$\Phi_{0.5}(t, [x]) = t\ell(\phi_{0.5}([x])) = t\ell(x).$$

Therefore we conclude that $\Phi_{0.5}^{-1} \circ \Phi_0 = \mathrm{id}$ at these points. But for $x \in (-0.5, 0)$

$$\Phi_0(t, [x]) = t\ell(\phi_0([x])) = t\ell(x)$$
$$\Phi_{0.5}(t, [x]) = t\ell(\phi_{0.5}([x])) = t\ell(\phi_{0.5}([x+1])) = t\ell(x+1) = -t\ell(x)$$

and so $\Phi_{0.5}^{-1} \circ \Phi_0 = -\mathrm{id}$ at these points.

(d) Suppose we have a non-vanishing global section s. Then in coordinates with respect to Φ_0 and $\Phi_{0.5}$ it is $\tilde{s}_0 : (-0.5, 0.5) \to \mathbb{R}$ and $\tilde{s}_{0.5} : (0, 1) \to \mathbb{R}$ respectively. Since it is non-vanishing, without loss of generality assume that $\tilde{s}_0 > 0$.

We know from the previous part that for $x \in (0, 0.5)$ that

$$\tilde{s}_{0.5}(x) = g_{U_{0.5},U_0}([x])\, \tilde{s}_0(x) = \tilde{s}_0(x) > 0.$$

On the other hand, for $x \in (0.5, 1)$

$$\tilde{s}_{0.5}(x) = g_{U_{0.5},U_0}([x])\, \tilde{s}_0(x-1) = -\tilde{s}_0(x-1) < 0.$$

Since $\tilde{s}_{0.5}$ is continuous, we see that $\tilde{s}_{0.5}(0.5) = 0$. But this contradicts the fact it is non-vanishing.

In Class Exercises

42. Isomorphism of bundles given as cocycles.

We have seen that cocycles are a convenient way to define a bundle. Suppose that we have two bundles E and E' over B given by cocycles g and g' on the same covering of B.

Suppose that there exist smooth functions $h_U : U \to \mathrm{GL}(F)$ for each open set U in the cover such that

$$g'_{V,U}(b) = h_V(b)\, g_{V,U}(b)\, h_U(b)^{-1}.$$

Show that the bundles E and E' are isomorphic.

Solution. Recall that bundles constructed by cocycle are defined as the quotient of trivial bundles $F \times U$. To construct an isomorphism between two such bundles, we will give the map on these trivialisations and show that it is well-defined with respect to the equivalence relation.

Call our map $G : E \to E'$. In the trivialisation over U we define $G : F \times U \subset E \to F \times U \subset E'$ by

$$G(f, b) := (h_U(b)f, b).$$

This is a bundle isomorphism in this trivialisation, because $h_U(b)$ is an invertible linear transformation by definition. Indeed, the inverse is

$$G^{-1}(f', b) = (h_U(b)^{-1}f', b) \in F \times U.$$

It only remains to show that this is a well-defined map.

If V is another open set of the cover and $b \in U \cap V$ then $(f_V, b) \sim (f_U, b)$ if and only if $f_V = g_{V,U}(b)f_U$ by definition of the equivalence relation on E. Now apply G to both of these

$$\begin{aligned} G(f_U, b) &= (h_U(b)f_U, b) \\ G(f_V, b) &= (h_V(b)f_V, b). \end{aligned}$$

Are these the same point in E'?

$$\begin{aligned} g'_{V,U}(b)(h_U(b)f_U) &= h_V(b)\, g_{V,U}(b)\, h_U(b)^{-1}\, h_U(b)f_U \\ &= h_V(b)\, g_{V,U}(b)f_U \\ &= h_V(b)f_V. \end{aligned}$$

So yes they are. Thus G is well-defined.

43. Direct sum of two Möbius bundles over $\mathbb{R}/\mathbb{Z}$.

Consider the Möbius bundle M over $\mathbb{R}/\mathbb{Z}$. Prove that $M \oplus M$ is the trivial rank 2 bundle over $\mathbb{R}/\mathbb{Z}$.

Solution. We will use the cocycle method we just proved. We have already computed the cocycles of M, so the cocycle of the direct sum is

$$g_{U_{0.5},U_0}([x]) = \begin{cases} I_2 & \text{for } x \in (0, 0.5) \\ -I_2 & \text{for } x \in (0.5, 1) \end{cases}$$

where I_2 is the 2-dimensional identity matrix. We need to find matrix function $h_0 : U_0 \to GL(\mathbb{R}^2)$ and $h_{0.5} : U_{0.5} \to GL(\mathbb{R}^2)$ that transform this cocycle into the trivial cocycle. Let $h : \mathbb{R} \to \mathrm{GL}(\mathbb{R}^2)$ be the half-speed rotation of the plane

$$h(x) = \begin{pmatrix} \cos \pi x & \sin \pi x \\ -\sin \pi x & \cos \pi x \end{pmatrix}.$$

Then we can take $h_0 = h \circ \phi_0$ and $h_{0.5} = h \circ \phi_{0.5}$. For $x \in (0, 0.5)$, $h_0([x]) = h(x) = h_{0.5}([x])$ and therefore

$$h_{0.5}([x])\, g_{U_{0.5},U_0}([x])\, h_0([x])^{-1} = h(x)\, I_2\, h(x)^{-1} = I_2.$$

However for $x \in (0.5, 1)$, $h_{0.5}([x]) = h(x)$ but

$$\begin{aligned} h_0([x]) = h_0([x-1]) = h(x-1) &= \begin{pmatrix} \cos \pi(x-1) & \sin \pi(x-1) \\ -\sin \pi(x-1) & \cos \pi(x-1) \end{pmatrix} \\ &= \begin{pmatrix} -\cos \pi x & -\sin \pi x \\ \sin \pi x & -\cos \pi x \end{pmatrix} = -h(x). \end{aligned}$$

In this case,

$$h_{0.5}([x])\, g_{U_{0.5},U_0}([x])\, h_0([x])^{-1} = h(x)\, (-I_2)\, (-h(x))^{-1} = I_2.$$

Whew. How can we understand this geometrically? We constructed the Möbius bundle originally as a line twisting in $\mathbb{R}^2$. We can imagine the second Möbius bundle as another line twisting in the same way, but at 90-degrees to the first one. This will also give a Möbius band. But the vector sum of the two lines is just all of $\mathbb{R}^2$.

44. Classification of line bundles over $\mathbb{R}/\mathbb{Z}$.

In this exercise we will classify all line bundles over $\mathbb{R}/\mathbb{Z}$. Let E be a line bundle over $\mathbb{R}/\mathbb{Z}$.

(a) Use a previous exercise to argue that $\pi^{-1}[U_0]$ and $\pi^{-1}[U_{0.5}]$ must be trivial bundles.

(b) Explain why $\mathrm{GL}(\mathbb{R}) = \mathbb{R}^\times$.

(c) Because E trivialisations over the cover $\mathcal{U} = \{U_0, U_{0.5}\}$, we can describe E over by a single cocycle $g = g_{U_{0.5},U_0} : U_0 \cap U_{0.5} \to \mathbb{R}^\times$. Let $\{\varphi_U : U \to [0, 1]\}$ be a partition of unity for $\mathcal{U}$. Define $h_U : U \to \mathbb{R}^+$ by

$$h_U([x]) = \left(\sum_{V \in \mathcal{U}} \varphi_V([x]) g_{V,U}([x])^2 \right)^{1/2}.$$

For example,

$$\begin{aligned} h_{U_0}([x]) &= \left(\varphi_{U_0}([x]) g_{U_0,U_0}([x])^2 + \varphi_{U_{0.5}}([x]) g_{U_{0.5},U_0}(x)^2 \right)^{1/2} \\ &= \left(\varphi_{U_0}([x]) + \varphi_{U_{0.5}}([x]) g([x])^2 \right)^{1/2}. \end{aligned}$$

Why is h_{U_0} well-defined at $[x] = [0]$ even though $g_{U_{0.5},U_0}([x])$ is not defined there? Show, using the cocycle properties that $h_{U_{0.5}} h_{U_0}^{-1} = |g|^{-1}$.

(d) Hence show that E is isomorphic to a bundle whose cocycle has $|g'_{U_{0.5},U_0}(x)| = 1$.

(e) Show that E is isomorphic to either the trivial bundle $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ or the Möbius bundle.

Solution.

(a) U_0 is diffeomorphic to the interval $(-0.5, 0.5)$, which is diffeomorphic to the whole real line. For example, using the map

$$t \mapsto \frac{\sin 2\pi t}{1+\cos 2\pi t}.$$

Since all line bundles over $\mathbb{R}$ are trivial (in fact all vector bundles over a contractible space are trivial and all line bundles over a simple connected space a trivial), E must be trivial over U_0.

(b) $GL(\mathbb{R})$ is the set of invertible linear maps from $\mathbb{R}$ to $\mathbb{R}$. Take a linear map $L : \mathbb{R} \to \mathbb{R}$ and call $L(1) = a$. For any other value $L(t) = tL(1) = ta$, which shows that L is just multiplication by a. If L is invertible, then $a \neq 0$. Thus $GL(\mathbb{R}) = \mathbb{R}^\times$ with group operation as multiplication and identity 1.

(c) There are a few things to notice with the definition of h_U. First, this is well-defined on all of U even though $g_{V,U}$ is only defined on the intersection $V \cap U$. This is because if $[x] \in U \setminus V$ then $\varphi_V([x]) = 0$ so we don't need to calculate $g_{V,U}([x])$ at those points. Further, the expression inside the bracket is a positive sum, so h_U is a positive function.

It's actually cleaner to prove the property in full generality:

$$\begin{aligned} h_U &= \left(\sum_{W\in\mathcal{U}} \varphi_W\, g_{W,U}^2\right)^{1/2} \\ &= \left(\sum_{W\in\mathcal{U}} \varphi_W \left(g_{W,V}\, g_{V,U}\right)^2\right)^{1/2} \\ &= \left(g_{V,U}^2\right)^{1/2} \left(\sum_{W\in\mathcal{U}} \varphi_W\, g_{W,V}^2\right)^{1/2} \\ &= |g_{V,U}| h_V. \end{aligned}$$

(d) By a previous exercise, E is isomorphic to a bundle with the cocycle

$$g'_{U_{0.5},U_0} = h_V g_{U_{0.5},U_0} h_U^{-1} = |g|^{-1} g.$$

This cocycle is unit length.

(e) We may as well assume that E is the represented by the cocycle from the previous part. $g_{U_{0.5},U_0}$ is a smooth function that takes the values ± 1. By taking $h_{U_0} =$

sign $g_{U_{0.5},U_0}([0.25])$ and $h_{U_{0.5}} = 1$ we can assume that $g_{U_{0.5},U_0}([0.25]) = 1$. There are only two such $g_{U_{0.5},U_0}$:

$$g_{U_{0.5},U_0}([x]) = 1,$$

and

$$g_{U_{0.5},U_0}([x]) = \begin{cases} 1 & \text{for } x \in (0, 0.5) \\ -1 & \text{for } x \in (0.5, 1) \end{cases}.$$

These are the trivial bundle and Möbius bundle respectively.

Additional Exercises

45. Triviality of the homomorphism bundle.

Let (E, B, π) and (E', B, π') be two vector bundles over a base manifold B. Consider the *homomorphism bundle* $(\mathrm{Hom}(E, E'), B, \pi'')$. People often say "Hom-bundle" for short. In parts (b) and (c) there are two methods of proof: try to find a non-vanishing sections or examine the cocycles.

(a) What is the rank of $\mathrm{Hom}(E, E')$.

(b) Show that when E and E' are trivial bundles, then so too is $\mathrm{Hom}(E, E')$.

(c) Prove or disprove: $\mathrm{Hom}(E, E')$ is trivial, then the bundles E and E' must be trivial. Hint: Examine the Möbius bundle M.

Solution.

(a) The rank is the dimension of the fibre. The fibres of the Hom-bundle are the homomorphisms from the fibre F of the first bundle to the fibre F' of the second. If these vector spaces are dimensions r and r' respectively then the homomorphisms can be identified with $r' \times r$ matrices. Hence they form a vector space of dimension rr'.

(b) If E and E' are trivial, we know that there exists non-vanishing sections $\{v_1, \ldots, v_r\}$ and $\{v'_1, \ldots, v'_{r'}\}$ which are every point are linearly independent. This means there is an isomorphism of vector bundles between E and $\mathbb{R}^r \times B$, and composing with the coordinate projections $\mathbb{R}^r \to \mathbb{R}$ gives us smooth functions $a_i : E \to \mathbb{R}$ such that $v = \sum_{k=1}^{r} a_k(v) v_k(\pi(v))$. These functions are linear because they are the composition of linear functions. Therefore we have bundle homomorphisms

$$s_{ij}(b) : v \mapsto a_i(v) v'_j(b) \in E' \text{ for } v \in \pi^{-1}[\{b\}].$$

The zero homomorphism is the one that maps all vectors to zero. Notice that for each of s_{ij} and for each $b \in B$ we have $s_{ij}(b)(v_i(b)) = v'_j(b) \neq 0$. Hence there is at least one vector in $\pi^{-1}[\{b\}]$ that is not mapped to zero, which shows that s_{ij} is non-vanishing.

They are also linearly independent: suppose that $0 = \sum_{k,l} c_{kl} s_{kl}(b)$. Applying this to the point $v_i(b)$ gives $0 = \sum_l c_{il} v'_l(b)$. The linear independence of the $v'_j(b)$ now forces $c_{il} = 0$ for all l. Repeating this with the other basis sections of E shows all coefficients to be zero.

We have found rr' linearly-independent non-vanishing sections of $\mathrm{Hom}(E, E')$. Therefore it is trivial.

Each of these functions is essentially the matrix with 1 in the $(i,j)^{th}$ component and 0 elsewhere because the map sends $v_i(b)$ to $v'_j(b)$ and other vectors $v_k(b)$ to zero and these are basis vectors of $\pi^{-1}[\{b\}]$ and $\pi'^{-1}[\{b\}]$ respectively. This proof was essentially the proof that matrices uniquely represent homomorphisms with respect to given bases of the vector spaces.

Now we give the proof in terms of the cocycles. If $g_{V,U}$ is a cocycle for E and $g'_{V,U}$ is a cocycle for E' then the cocycle for the hom bundle is complicated. It is a map $\tilde{g}_{V,U} : V \cap U \to GL(Hom(E, E'))$ that acts by

$$C \mapsto g'_{V,U} \circ C \circ g^{-1}_{V,U}.$$

But if both bundles are trivial then they can be represented by cocycles that are the identity. Then $\tilde{g} : C \to C$ is also the identify.

(c) This is false. We give as our counterexample the bundle $H := \mathrm{Hom}(M, M)$ as suggested by the hint. Because the rank of M is 1, so too is the rank of H, as discussed in part (a). Thus it is sufficient to give a non-vanishing section of H. But this is easy: the identity map id_M fits the description.

Let us give a more complete picture of H. Every bundle homomorphism $s : M \to M$ must act as scalar multiplication on each fibre, because those are the only homomorphism $\mathbb{R} \to \mathbb{R}$. But scaling the fibre is independent of the choice of trivialisations; the trivialisations preserve the vector space structure. Therefore for each point $h \in H$ we can describe it as a pair (a, x) where $x = \pi(h)$ is the base point and a is the scalar. Conversely, given (a, x) consider the homomorphism $a\mathrm{id}$ on $\pi^{-1}[\{x\}]$. This describes the correspondence between H and the trivial bundle $\mathbb{R} \times \mathbb{S}^1$.

This proof does not generalise to higher dimensional homomorphism bundles, because in general there are many more homomorphisms $\mathbb{R}^r \to \mathbb{R}^{r'}$ than just scaling, and these other homomorphisms do not need to be preserved by the trivialisations. It does not even generalise to the Hom-bundle between line bundles L and L', because while it is true that $\mathrm{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}^\times$, how to identify the fibres of L and L'

with $\mathbb{R}$ depends on the trivialisations. It does generalise to the bundle $\mathrm{Hom}(L, L)$ for (L, B, π) a line bundle, because then we can use the special homomorphism id.

We have seen in the previous part that the cocycles of a Hom bundle can be complicated. However in one-dimension, all linear maps commute (they are just multiplication). So the cocycle $\tilde{g}_{V,U}$ is just

$$C \mapsto g_{V,U} \circ C \circ g_{V,U}^{-1} = C.$$

In other words, $\tilde{g}_{V,U} = 1$. Hence it is trivial.

46. The dual bundle of a vector bundle.

Let (E, B, π) be a vector bundle over a manifold B, with fibre $F = \mathbb{R}^n$. Further, let $\mathcal{U}$ be an open cover of B so that π trivialises over every set $U \in \mathcal{U}$. Denote the cocycles of π with respect to this cover by $g_{V,U} : U \cap V \to \mathrm{GL}(\mathbb{R}^n)$.

Show that the *dual bundle* $(\tilde{E}, B, \tilde{\pi})$ to π is described over $\mathcal{U}$ by the cocycle $(\tilde{g}_{V,U})_{V,U \in \mathcal{U}}$ with

$$\tilde{g}_{V,U} : U \cap V \to \mathrm{GL}(\mathbb{R}^n),\ x \mapsto \big(g_{V,U}(x)^T\big)^{-1}.$$

Solution. The dual bundle is by definition a special type of homomorphism bundle, namely $\mathrm{Hom}(E, \mathbb{R} \times B)$. Thus we should look to Theorem 1.59. In that theorem, the cocycle of a Hom-bundle is described by the function $\Pi(A, B) : C \mapsto B \circ C \circ A^{-1} \in \mathrm{Hom}(F, F')$ where $A \in \mathrm{GL}(F)$ and $B \in \mathrm{GL}(F')$ are respectively the transition functions of the source and target bundles at a point and $C \in \mathrm{Hom}(F, F')$ is a homomorphism between the fibres F and F' and thus itself a point of the fibre of the Hom-bundle.

In this situation, we have $F = \mathbb{R}^n$ and $F' = \mathbb{R}$, and at some point $b \in U \cap V \subset B$ we have $A = g_{V,U}(b) \in \mathrm{GL}(\mathbb{R}^n)$ and $B = 1$ (the transition functions for the trivial bundle are always the identity matrix). We can also describe $C \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ as a column vector that acts by $C : v \mapsto C^T v$ (we will explain why we should think of it this way shortly). Thus, for $v \in \mathbb{R}^n$ the transition acts as

$$\begin{aligned}
\tilde{g}_{V,U}(b)(C) &: v \mapsto B(C(A^{-1}(v))) = C^T g_{V,U}(b)^{-1} v = \big((g_{V,U}(b)^T)^{-1} C\big)^T v \in \mathbb{R} \\
\tilde{g}_{V,U}(b) &: C \mapsto (g_{V,U}(b)^T)^{-1} C \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) \\
\tilde{g}_{V,U}(b) &= (g_{V,U}(b)^T)^{-1} \in \mathrm{GL}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}))
\end{aligned}$$

We can now explain why we thought of $C \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ as a column vector, because we want it to be acted on by an element of $\mathrm{GL}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}))$ and these act on column vectors. Notice that transpose and inversion of matrices commute, so it doesn't matter which order we write those operations.