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Analysis III
7. Exercise: Vector Bundles II

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Preparation Exercises

41. A non-trivial line bundle over $\mathbb{R}/\mathbb{Z}$.

Consider the lines $L_x = \mathbb{R}(\cos \pi x, \sin \pi x) \subset \mathbb{R}^2$. Notice that $L_{x+1} = L_x$.

(a) By writing $M := \{(v, [x]) \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \mid v \in L_x\}$ locally as the level set of functions $F : \mathbb{R}^2 \times U_x \to \mathbb{R}$, prove it is a submanifold.

Hint. How can you write the line L_x as the zero set of a function?

(b) Show M is a vector bundle over $\mathbb{R}/\mathbb{Z}$. It is called the *Möbius band* or *Möbius bundle.*

Hint. Can you find non-vanishing local sections?

(c) Compute the cocycle $g_{U_{0.5},U_0}$.

(d) Prove that $(M, \mathbb{R}/\mathbb{Z}, \pi)$ is a non-trivial bundle.

Hint. Prove there are no non-vanishing global sections.

In Class Exercises

42. Isomorphism of bundles given as cocycles.

We have seen that cocycles are a convenient way to define a bundle. Suppose that we have two bundles E and E' over B given by cocycles g and g' on the same covering of B.

Suppose that there exist smooth functions $h_U : U \to \mathrm{GL}(F)$ for each open set U in the cover such that

$$g'_{V,U}(b) = h_V(b)^{-1}\, g_{V,U}(b)\, h_U(b).$$

Show that the bundles E and E' are isomorphic.

43. Direct sum of two Möbius bundles over $\mathbb{R}/\mathbb{Z}$.

Consider the Möbius bundle M over $\mathbb{R}/\mathbb{Z}$. Prove that $M \oplus M$ is the trivial rank 2 bundle over $\mathbb{R}/\mathbb{Z}$.

44. Classification of line bundles over $\mathbb{R}/\mathbb{Z}$.

In this exercise we will classify all line bundles over $\mathbb{R}/\mathbb{Z}$. Let E be a line bundle over $\mathbb{R}/\mathbb{Z}$.

(a) Use a previous exercise to argue that $\pi^{-1}[U_0]$ and $\pi^{-1}[U_{0.5}]$ must be trivial bundles.

(b) Explain why $\mathrm{GL}(\mathbb{R}) = \mathbb{R}^\times$.

(c) Because E trivialisations over the cover $\mathcal{U} = \{U_0, U_{0.5}\}$, we can describe E over by a single cocycle $g = g_{U_{0.5},U_0} : U_0 \cap U_{0.5} \to \mathbb{R}^\times$. Let $\{\varphi_U : U \to [0,1]\}$ be a partition of unity for $\mathcal{U}$. Define $h_U : U \to \mathbb{R}^+$ by

$$h_U([x]) = \left(\sum_{V \in \mathcal{U}} \varphi_V([x]) g_{V,U}([x])^2 \right)^{-1/2}.$$

For example,

$$\begin{aligned} h_{U_0}([x]) &= \left(\varphi_{U_0}([x]) g_{U_0,U_0}([x])^2 + \varphi_{U_{0.5}}([x]) g_{U_{0.5},U_0}(x)^2\right)^{-1/2} \\ &= \left(\varphi_{U_0}([x]) + \varphi_{U_{0.5}}([x]) g([x])^2\right)^{-1/2}. \end{aligned}$$

Why is h_{U_0} well-defined at $[x] = [0]$ even though $g_{U_{0.5},U_0}([x])$ is not defined there? Show, using the cocycle properties that $h_{U_{0.5}}^{-1} h_{U_0} = |g|^{-1}$.

(d) Hence show that E is isomorphic to a bundle whose cocycle has $|g'_{U_{0.5},U_0}(x)| = 1$.

(e) Show that E is isomorphic to either the trivial bundle $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ or the Möbius bundle.

Additional Exercises

45. Triviality of the homomorphism bundle.

Let (E, B, π) and (E', B, π') be two vector bundles over a base manifold B. Consider the *homomorphism bundle* $(\mathrm{Hom}(E, E'), B, \pi'')$. People often say "Hom-bundle" for short. In parts (b) and (c) there are two methods of proof: try to find a non-vanishing sections or examine the cocycles.

(a) What is the rank of $\mathrm{Hom}(E, E')$.

(b) Show that when E and E' are trivial bundles, then so too is $\mathrm{Hom}(E, E')$.

(c) Prove or disprove: $\mathrm{Hom}(E, E')$ is trivial, then the bundles E and E' must be trivial.

Hint: Examine the Möbius bundle M.

46. The dual bundle of a vector bundle.

Let (E, B, π) be a vector bundle over a manifold B, with fibre $F = \mathbb{R}^n$. Further, let $\mathcal{U}$ be an open cover of B so that π trivialises over every set $U \in \mathcal{U}$. Denote the cocycles of π with respect to this cover by $g_{V,U} : U \cap V \to \mathrm{GL}(\mathbb{R}^n)$.

Show that the *dual bundle* $(\tilde{E}, B, \tilde{\pi})$ to π is described over $\mathcal{U}$ by the cocycle $(\tilde{g}_{V,U})_{V,U \in \mathcal{U}}$ with

$$\tilde{g}_{V,U} : U \cap V \to \mathrm{GL}(\mathbb{R}^n),\ x \mapsto \left(g_{V,U}(x)^T\right)^{-1}.$$