Martin Schmidt
Ross Ogilvie

Analysis III
6. Exercise: Vector Bundles

17 October 2022

Preparation Exercises

34. Adapted charts for vector bundles.

Let (E, B, π) be a $\mathbb{R}$-vector bundle with fibre F. Let $U \subset B$ be an open set. By shrinking U if necessary, we can assume that U is the domain of chart ϕ of B and that E trivialises over U, in the sense that $\Phi : F \times U \to \pi^{-1}[U]$ is a local trivialisation.

(a) Why is $\pi^{-1}[U]$ an open set of E?

(b) Prove that $\psi = (\mathrm{id}_F \times \phi) \circ \Phi^{-1} : \pi^{-1}[U] \to F \times U \to F \times \phi[U]$ by which I mean

$$\psi : e \mapsto (\tilde{\Phi}(v), \pi(e)) = \Phi^{-1}(e) \mapsto (\tilde{\Phi}(v), \phi(\pi(e)))$$

is a compatible chart for E. We call these charts of E adapted to the bundle structure.

(c) Recall the definition of a section of a vector bundle.

(d) A local section s over U is a map $U \to \pi^{-1}[E]$ between manifolds. Show that

$$s(b) = \Phi(\tilde{s}(b), b)$$

for $\tilde{s} : U \to F$. This is called writing a section with respect to the trivialisation Φ.

(e) Prove that s is smooth if an only if $\tilde{s}$ is smooth.

Solution.

(a) A local trivialisation is a diffeomorphism $\Phi : F \times U \to \pi^{-1}[U]$ (Definition 1.49). In particular it is an open map, and so $\pi^{-1}[U] = \Phi[F \times U]$ is open.

(b) Let us explain ψ. The inverse of Φ is also a smooth map $\Phi^{-1} : \pi^{-1}[U] \to F \times U$. In components, $\Phi^{-1}(e) = (\Phi_1^{-1}(e), \Phi_2^{-1}(e))$. Both components must be smooth functions, since Φ^{-1} is smooth. But Φ must also agree with the bundle projection in that $\pi \circ \Phi = p_2$. Precomposing with Φ^{-1} gives $\pi = p_2 \circ \Phi^{-1} = \Phi_2^{-1}$. Using the notation $\tilde{\Phi} = \Phi_1^{-1}$ gives us $\Phi^{-1}(e) = (\tilde{\Phi}(e), \pi(e))$. Now applying the smooth map ϕ to the second component gives us ψ.

In terms of proving the exercise, ψ is the composition of smooth maps, and it has a smooth inverse $\Phi \circ (\mathrm{id}_F \times \phi^{-1})$. Therefore it is a diffeomorphism to $F \times \phi[U] \subset F \times \mathbb{R}^n$. Since F is a real vector space, it is basically $\mathbb{R}^m$. Therefore we have a diffeomorphism to a subset of Euclidean space, ie a chart. (Recall, one way to think of charts are that they are diffeomorphisms of open subsets of the manifold to open subsets of Euclidean space.)

(c) A section $s : B \to E$ is a smooth map with the property that $\pi \circ s = \mathrm{id}_B$.

(d) Let $\Phi^{-1} \circ s(b) = (\tilde{s}(b), s_2(b))$. By the section property,

$$s_2 = p_2 \circ \Phi^{-1} \circ s = \pi \circ s = \mathrm{id}_B.$$

Since s is a map between manifolds, we could go further and write s in coordinates with respect to ϕ on B and ψ on E. Therefore

$$\begin{aligned} \psi \circ s \circ \phi^{-1}(y) &= (\mathrm{id}_F \times \phi) \circ \Phi^{-1} \circ s \circ \phi^{-1}(y) \\ &= (\mathrm{id}_F \times \phi)(\tilde{s}(\phi^{-1}(y)), \phi^{-1}(y)) \\ &= (\tilde{s}(\phi^{-1}(y)), y). \end{aligned}$$

The nice part about using the chart ψ on E is that all the information of the section is contained in $\tilde{s}$, the second component is just a formality.

(e) Clearly s is smooth if and only if $\tilde{s}$ is, since Φ is a diffeomorphism.

35. The tangent bundle.

Let's examine Theorem 1.54.

Let $f : X \to Y$. We have seen the tangent map $T_x(f) : T_xX \to T_{f(x)}Y$ at a point x. The tangent map $T(f)$ is a map from the tangent bundle of X

$$TX = \bigcup_{x \in X} T_xX = \{(v, x) \mid x \in X, v \in T_xX\}$$

to the tangent bundle of Y. Though technically unnecessary, it is often useful to write points of TX as pairs. The tangent map then acts as

$$(v, x) \mapsto (T_x(f)(v), f(x)).$$

In Theorem 1.54 we see how to use the tangent maps of charts $T(\phi)$ are charts for the tangent bundle.
Explain what π is for the tangent bundle.
What are the local trivialisations for a tangent bundle?
Show that the cocycles of a tangent bundle are the same as the change of coordinates for tangent vectors.

Solution. Every tangent vector belongs to a tangent space T_xX at a particular point x. π is the map that takes a tangent vector and tells you which point it belong to. If a tangent vector is represented by a curve, $\pi([\alpha]) = \alpha(0)$.

The local trivialisations for a tangent bundle are nothing other than $(T(\phi))^{-1} = T(\phi^{-1}) : \mathbb{R}^n \times U \to TU$. This maps (v, x) to $([\alpha_{v,x}], x)$ for $\alpha_{v,x}(t) = \phi^{-1}(tv + \phi(x))$ a curve through x.

We use the two charts $\phi_1 : U_1 \to \mathbb{R}$ and $\phi_2 : U_2 \to \mathbb{R}$. These give us local trivialisations Φ_1, Φ_2 of the tangent bundle. If we compose local trivialisations together we get a map

$$\Phi_2^{-1} \circ \Phi_1 : \mathbb{R}^n \times (U_1 \cap U_2) \to T(U_1 \cap U_2) \to \mathbb{R}^n \times (U_1 \cap U_2).$$

We can understand this using the special forms from the previous exercise.

$$\Phi_2^{-1} \circ \Phi_1(v, x) = \Phi_2^{-1}\Big(([\alpha_{1,v,x}], x)\Big) = \Big(\tilde{\Phi}_2([\alpha_{1,v,x}]), x\Big)$$

We see that the second part is just the identity map. Therefore the information is all contained in the first component. For each x we get a map $v \mapsto \tilde{\Phi}_2([\alpha_{1,v,x}])$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. This is a matrix in $GL(\mathbb{R}^n)$. The map from x to this matrix is the cocycle, or more properly one element of the cocycle (the cocycle is the collection of all of these).

So let's calculate it. $\Phi_2^{-1} = (T(\phi_2)^{-1})^{-1} = T(\phi_2)$. What is $\tilde{\Phi}_2$?

$$\begin{aligned}\Phi_2^{-1}\Big(([\alpha_{1,v,x}], x)\Big) &= T(\phi_2)\Big(([\alpha_{1,v,x}], x)\Big) \\ &= \Big(T_x(\phi_2)([\alpha_{1,v,x}]), x\Big)\end{aligned}$$

We see $\tilde{\Phi}_2$ is the tangent map at a point.

$$\begin{aligned}\tilde{\Phi}_2([\alpha_{1,v,x}]) = T_x(\phi_2)([\alpha_{1,v,x}]) &= J_0(\phi_2 \circ \phi_1^{-1}(tv + \phi_1(x))) \\ &= J_{\phi_1(x)}(\phi_2 \circ \phi_1^{-1}) J_0(tv + \phi_1(x)) \\ &= J_{\phi_1(x)}(\phi_2 \circ \phi_1^{-1}) v.\end{aligned}$$

This shows us that the cocycle for a tangent bundle is nothing other than the change of coordinates for vectors.

In Class Exercises

36. Non-vanishing sections and local trivialisations.

For a line bundle (a vector bundle with rank 1) there is a correspondence between non-vanishing sections and local trivialisations. What is it?

Solution. Let $\Phi : \mathbb{R} \times U \to \pi^{-1}[U]$ be a local trivialisation. Then $s(b) = \Phi(1, b)$ is a non-vanishing section over U.

Conversely, suppose that s is a non-vanishing section over U. Because $s(x)$ is not zero, it spans $\pi^{-1}[\{x\}]$. Therefore every element of the fibre is $ts(x)$ for some $t \in \mathbb{R}$. This gives the local trivialisation $\Phi(t,x) = ts(x)$ over U. To see this is actually a trivialisation requires showing Φ is smooth, but this follows from Exercise 38(b).

37. The tangent bundles of low dimensional spheres.

In this exercise we will examine the tangent bundle of the n-sphere

$$\mathbb{S}^n := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \ldots + x_{n+1}^2 = 1 \},$$

for $n \leq 3$. We have seen previously that we can make the identification

$$T_x\mathbb{S}^n = \{ w \in \mathbb{R}^{n+1} \,|\, w \cdot x = 0 \} \,.$$

This means that we can describe a section of $T\mathbb{S}^n$ as a smooth function $s : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $s(x) \cdot x = 0$ for all $x \in \mathbb{S}^n$.

(a) Find a non-vanishing section of the tangent bundle $T\mathbb{S}^1$ (a section that never takes the value 0).

Hence $T\mathbb{S}^1$ is trivial. *(3 Points)*

(b) Show that the vector bundle $T\mathbb{S}^3$ is trivial. *(2 Points)*

Hint. Use Lemma 1.58 and consider the following sections

$$f_1(x_1, x_2, x_3, x_4) := (-x_2, x_1, x_4, -x_3)\,, \quad f_2(x_1, x_2, x_3, x_4) := (-x_3, -x_4, x_1, x_2)$$
$$\text{and} \quad f_3(x_1, x_2, x_3, x_4) := (-x_4, x_3, -x_2, x_1)$$

Remark. We can identify $\mathbb{S}^3$ with the unit sphere in the Quaternions $\mathbb{H}$. Then $f_1 = ix$, $f_2 = jx$ and $f_3 = kx$.

(c) Let $x_N := (1,0,0) \in \mathbb{S}^2$ and $x_S := (-1,0,0) \in \mathbb{S}^2$. With the aid of stereographic projection ϕ_N and ϕ_S, write down local trivialisations of $T\mathbb{S}^2$ over $U_N := \mathbb{S}^2 \setminus \{x_N\}$ and $U_S := \mathbb{S}^2 \setminus \{x_S\}$, and calculate the transition function $g_{U_N,U_S} : \mathbb{S}^2 \setminus \{x_N, x_S\} \to \mathrm{GL}(\mathbb{R}^2)$. *(8 Points)*

Remark. $T\mathbb{S}^2$ is not trivial, but this require some more theory to prove. It is a consequence of the "hairy ball theorem": every global section of $T\mathbb{S}^2$ has a zero.

Solution.

(a) In $\mathbb{R}^2$ there is the rotation operator $R(x,y) = (-y,x)$. This creates an equal-length perpendicular vector, ie $|x| = |R(x)|$ and $x \cdot R(x) = 0$. The section $x \mapsto (R(x), x)$ is a section of the tangent bundle and non-vanishing.

(b) First, note the value of these functions are perpendicular to x, eg $(x_1, x_2, x_3, x_4) \cdot (-x_2, x_1, x_4, -x_3) = -x_1x_2 + x_2x_1 + x_3x_4 - x_4x_3 = 0$, and unit length $|(-x_2, x_1, x_4, -x_3)| = |x| = 1$. Hence they are non-vanishing sections of $T\mathbb{S}^3$. It remains to show they are linearly independent. But this follows from the fact that they are all perpendicular, eg

$$f_1 \cdot f_2 = (-x_2, x_1, x_4, -x_3) \cdot (-x_3, -x_4, x_1, x_2) = x_2x_3 - x_1x_4 + x_4x_1 - x_3x_2 = 0.$$

Hence by Lemma 1.58 it follows that $T\mathbb{S}^3$ is trivial.

(c) Before we jump into calculation, let us simplify first. The identification of the tangent space with a subset of Euclidean space is formally the tangent map of the inclusion $T(\iota)$. The trivialisation coming from ϕ_S is $T(\phi_S^{-1})$. Therefore we want to calculate $T(\iota) \circ T(\phi_S^{-1}) = T(\iota \circ \phi_S^{-1})$. But the tangent map of a map between Euclidean spaces is just the Jacobian.

$$\iota \circ \phi_S^{-1}(y) = \frac{1}{1+|y|^2}\left(1 - |y|^2, 2y_1, 2y_2\right) = \frac{1}{1+y_1^2+y_2^2}\left(1 - y_1^2 - y_2^2, 2y_1, 2y_2\right)$$

$$T(\iota \circ \phi_S^{-1}) = \frac{1}{(1+|y|^2)^2}\begin{pmatrix} -2y_1(1+|y|^2) - (1-|y|^2)2y_1 & -2y_2(1+|y|^2) - (1-|y|^2)2y_2 \\ 2(1+|y|^2) - 2y_12y_1 & -2y_22y_1 \\ -2y_22y_1 & 2(1+|y|^2) - 2y_22y_2 \end{pmatrix}$$

$$= \frac{1}{(1+|y|^2)^2}\begin{pmatrix} -4y_1 & -4y_2 \\ 2(1-y_1^2+y_2^2) & -4y_1y_2 \\ -4y_1y_2 & 2(1+y_1^2-y_2^2) \end{pmatrix}.$$

The trivialisation from ϕ_N is similar.

As we have seen in a previous exercise, the fact that the local trivialisations are $T(\phi_{U_N}^{-1})$ and $T(\phi_{U_S}^{-1})$ gives

$$g_{U_N,U_S} = (T(\phi_{U_S}^{-1}))^{-1} \circ T(\phi_{U_N}^{-1}) = T(\phi_{U_S}) \circ T(\phi_{U_N}^{-1}) = T(\phi_{U_S} \circ \phi_{U_N}^{-1}).$$

The transition between charts $\phi_{U_S} \circ \phi_{U_N}^{-1}$ is simply $y \mapsto \|y\|^{-2}\, y$. And this is a map between Euclidean spaces, so the tangent map is just Jacobian and we calculate as normal:

$$J(\phi_{U_S} \circ \phi_{U_N}^{-1}(y)) = \frac{1}{\|y\|^4}\begin{pmatrix} y_2^2 - y_1^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix}.$$

We should write it not in local coordinates $y \in \mathbb{R}^2$ but rather in terms of $x \in \mathbb{S}^2$, with $y = \phi_N(x)$. Then

$$g_{U_N,U_S}(x) = \frac{1}{(1+x_0)^2}\begin{pmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix}.$$

The fact that the entries of the matrix look like the real and imaginary parts of $(x_1 + ix_2)^2$ is not a coincidence, but the explanation belongs to the theory of complex manifolds.

Additional Exercises

38. Sections of vector bundles.

Let (E, B, π) be a $\mathbb{R}$-vector bundle with fibre F, $s, s_1, s_2 : B \to E$ smooth sections of (E, B, π), and $f : B \to \mathbb{R}$ a smooth function. Prove:

(a) The *zero section* $O : B \to E$, $b \mapsto 0_b$ is a smooth section. By 0_b we mean this: $F_b = \pi^{-1}[\{b\}]$ is a vector space, so it has a zero element $0_b \in F_b \subset E$. *(2 Points)*

(b) $s_1 + s_2$ and $g \cdot s$ are smooth sections. *(2 Points)*

(c) Interpret f as a global section of the trivial bundle $E = \mathbb{R} \times B$. *(1 Point)*

(d) The image $s[B]$ is a submanifold of E. *(3 Points)*

Solution.

(a) We have already developed a criterion to show something is a smooth section, namely to write it in a special chart of the bundle. In these charts the zero section is $b \mapsto \Phi(0, b)$. This is clearly smooth.

(b) First, just consider: $s_1(b)$ and $s_2(b)$ are points of the manifold E. What does it even mean to add together points of a manifold?!

This is why it is important that s_1, s_2 are sections and E is a vector bundle. Because they are sections: $s_1(b), s_2(b) \in \pi^{-1}[\{b\}]$. Because it is a vector bundle: $\pi^{-1}[\{b\}]$ is a vector space. This is the reason these operations make sense. We say that they are defined *fibre-wise*.

We can use any local trivialisation to give the sum.

$$(s_1 + s_2)(b) = \Phi(\tilde{s}_1(b) + \tilde{s}_2(b), b).$$

This is smooth. Likewise

$$(f \cdot s)(b) := (f(b) \cdot \tilde{s}(b), b)$$

is a smooth section.

(c) More generally from a global function $\tilde{h} : B \to F$ we get a section $b \mapsto (\tilde{h}(b), b)$ of the trivial bundle $F \times B$. Conversely, a section of this bundle gives a function through projection to the first component.

This does *not* work for any bundle, because in general we only have projection to the second component and this just gives back the point in B. Projection to the first component depends on the trivialisation Φ. For this reason, sections are a generalisation of functions from a manifold B to a vector space.

(d) First observe that $\pi \circ s = \mathrm{id}_B$ shows that s is a homeomorphism. It remains to show that it an immersion. But we have already seen that a section in local coordinates has the form $\psi \circ s \circ \phi^{-1}(x) = (\tilde{s} \circ \phi^{-1}(x), x)$, so the Jacobian is $(J(\tilde{s} \circ \phi^{-1}) \,|\, \mathbb{1})$. The identity matrix block shows that it is injective.

In fact we can say something even stronger: s is a diffeomorphism between B and $s(B)$. In particular, since in every vector bundle the zero section is a global section, it is common to identity the image of the zero section in E with the base manifold B.

39. The tangent bundle of a product.

Why is $T(X \times Y) = TX \times TY$. Consult Definition 1.41 and try to write $T_{(x,y)}(X \times Y)$ as a product.

Solution. Let (ϕ, U) be a chart of X and (ψ, V) be a chart of Y. By Definition 1.41 we have that $(\phi \times \psi, U \times V)$ is a chart of $X \times Y$. Consider the tangent space of the product. Choose any vector in $T_{(x,y)}(X \times Y)$ represented by $\alpha = (\alpha_X, \alpha_Y) : (-\varepsilon, \varepsilon) \to X \times Y$. Then we get vectors $[\alpha_X] \in T_x X$ and $[\alpha_Y] \in T_y Y$ given by the two components of α. Conversely, given vectors in $T_x X$ and $T_y Y$ we can make a path in $X \times Y$ and get a vector in $T_{(x,y)}(X \times Y)$. Moreover, if $\beta = (\beta_X, \beta_Y)$ is another path, then $[\alpha] = [\beta]$ if and only if

$$\begin{aligned} ((\phi \times \psi) \circ \alpha)'(0) &= ((\phi \times \psi) \circ \beta)'(0) \\ (\phi \circ \alpha_X, \psi \circ \alpha_Y)'(0) &= (\phi \circ \beta_X, \psi \circ \beta_Y)'(0) \\ ((\phi \circ \alpha_X)'(0), (\psi \circ \alpha_Y)'(0)) &= ((\phi \circ \beta_X)'(0), (\psi \circ \beta_Y)'(0)), \end{aligned}$$

which is exactly the condition that $[\alpha_X] = [\beta_X]$ and $[\alpha_Y] = [\beta_Y]$. Hence we see that $T_{(x,y)}(X \times Y) = T_x X \times T_y Y$.

This splitting of the tangent space of $X \times Y$ allows us to write projection maps $p_X : T(X \times Y) \to TX$ and $p_Y : T(X \times Y) \to TY$. It is a short exercise to check that this makes $T(X \times Y)$ into a product manifold.

40. Line bundles over $\mathbb{R}$ are trivial. Prove that every line bundle (a vector bundle whose fibre dimension is 1) over $\mathbb{R}$ is trivial. *(8 Points)*

Hint. Let $(E, \mathbb{R}, \pi)$ be a line bundle. Choose a point x_0 and show that there is an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ with a non-vanishing section s. Then consider

$$J := \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{There exists an extension } s_x \text{ of } s \text{ to } (x, x_0 + \varepsilon) \text{ or } (x_0 - \varepsilon, x), \\ \text{such that } s_x \text{ is non-vanishing} \end{array} \right\},$$

where the choice of $(x, x_0 + \varepsilon)$ or $(x_0 - \varepsilon, x)$ depends whether $x \le x_0$ or $x \ge x_0$ Show that J is non-empty, open. Argue further that $J = \mathbb{R}$.

Solution. We might as well take $x_0 = 0$. There is a trivialisation Φ_U over $U \ni 0$. We may assume that $U = (-\varepsilon, \varepsilon)$. Consider the local section s over U given by $x \mapsto \Phi_U(1, x) \in E$. This is non-vanishing.

With s in hand, we can now define J. Immediately $(-\varepsilon, \varepsilon) \subset J$ so it is non-empty. Choose $x \in J$ with $x > 0$. Let $\Phi_{U'}$ be a local trivialisation containing x. Again we can assume that $U' = (x - \eta, x + \eta)$ with $x - \eta > 0$. By the definition of J, we know there exists $s_x(y) = \Phi_{U'}(\tilde{s}_x(y), y)$ for $y \in (-\varepsilon, x) \cap (x - \eta, x + \eta) = (x - \eta, x)$ with $\tilde{s}_x(y)$ a non-vanishing smooth function. We can extend this a non-vanishing function f on all of $(x - \eta, x + \eta)$. Then

$$\begin{cases} s_x(y) & \text{for } y \in (-\varepsilon, x) \\ \Phi_{U'}(f(y), y) & \text{for } y \in (x - \eta, x + \eta) \end{cases}$$

is a smooth non-vanishing section on $(-\varepsilon, x + \eta)$. This shows that J is open at x. Together with a similar proof for $x < 0$ we get that J is open.

Suppose that there were a point $x \notin J$ with $x > 0$. Because J is open $[0, x+1] \cap (\mathbb{R}^+ \setminus J)$ is compact. Thus there is a minimum point $x \notin J$ with $x > 0$. Choose an interval $(x - \eta, x + \eta)$ over which E trivialises. Since x is minimal there must exist a non-vanishing section s_t over $(-\varepsilon, t)$ for $t \in (x - \eta, x)$. But then we can extend s_t to $(-\varepsilon, x + \eta)$ in the same way as above. Therefore $x \in J$, which is a contradiction. A similar argument shows that all negative points belong to J as well. This completes the proof that $J = \mathbb{R}$.

Terminology

Schnitt = section
nullstellenfreien = non-vanishing
Geradenbündel = line bundle
American spelling is fiber, British spelling is fibre.