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# Analysis III
# 6. Exercise: Vector Bundles

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## Preparation Exercises

### 34. Adapted charts for vector bundles.

Let $(E, B, \pi)$ be a $\mathbb{R}$-vector bundle with fibre $F$. Let $U \subset B$ be an open set. By shrinking $U$ if necessary, we can assume that $U$ is the domain of chart $\phi$ of $B$ and that $E$ trivialises over $U$, in the sense that $\Phi : F \times U \to \pi^{-1}[U]$ is a local trivialisation.

**(a)** Why is $\pi^{-1}[U]$ an open set of $E$?

**(b)** Prove that $\psi = (\mathrm{id}_F \times \phi) \circ \Phi^{-1} : \pi^{-1}[U] \to F \times U \to F \times \phi[U]$ by which I mean

$$\psi : e \mapsto (\tilde{\Phi}(v), \pi(e)) = \Phi^{-1}(e) \mapsto (\tilde{\Phi}(v), \phi(\pi(e)))$$

is a compatible chart for $E$. We call these charts of $E$ adapted to the bundle structure.

**(c)** Recall the definition of a section of a vector bundle.

**(d)** A local section $s$ over $U$ is a map $U \to \pi^{-1}[E]$ between manifolds. Show that

$$s(b) = \Phi(\tilde{s}(b), b)$$

for $\tilde{s} : U \to F$. This is called writing a section with respect to the trivialisation $\Phi$.

**(e)** Prove that $s$ is smooth if an only if $\tilde{s}$ is smooth.

### 35. The tangent bundle.

Let's examine Theorem 1.54.

Let $f : X \to Y$. We have seen the tangent map $T_x(f) : T_xX \to T_{f(x)}Y$ at a point $x$. The tangent map $T(f)$ is a map from the tangent bundle of $X$

$$TX = \bigcup_{x \in X} T_xX = \{(v, x) \mid x \in X, v \in T_xX\}$$

to the tangent bundle of $Y$. Though technically unnecessary, it is often useful to write points of $TX$ as pairs. The tangent map then acts as

$$(v, x) \mapsto (T_x(f)(v), f(x)).$$

In Theorem 1.54 we see how to use the tangent maps of charts $T(\phi)$ are charts for the tangent bundle.
Explain what $\pi$ is for the tangent bundle.
What are the local trivialisations for a tangent bundle?
Show that the cocycles of a tangent bundle are the same as the change of coordinates for tangent vectors.

## In Class Exercises

**36. Non-vanishing sections and local trivialisations.**

For a line bundle (a vector bundle with rank 1) there is a correspondence between non-vanishing sections and local trivialisations. What is it?

**37. The tangent bundles of low dimensional spheres.**

In this exercise we will examine the tangent bundle of the $n$-sphere

$$\mathbb{S}^n := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \ldots + x_{n+1}^2 = 1 \},$$

for $n \leq 3$. We have seen previously that we can make the identification

$$T_x\mathbb{S}^n = \{ w \in \mathbb{R}^{n+1} \,|\, w \cdot x = 0 \} .$$

This means that we can describe a section of $T\mathbb{S}^n$ as a smooth function $s : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $s(x) \cdot x = 0$ for all $x \in \mathbb{S}^n$.

**(a)** Find a non-vanishing section of the tangent bundle $T\mathbb{S}^1$ (a section that never takes the value 0).

Hence $T\mathbb{S}^1$ is trivial. *(3 Points)*

**(b)** Show that the vector bundle $T\mathbb{S}^3$ is trivial. *(2 Points)*

Hint. Use Lemma 1.58 and consider the following sections

$$f_1(x_1, x_2, x_3, x_4) := (-x_2, x_1, x_4, -x_3) \,, \quad f_2(x_1, x_2, x_3, x_4) := (-x_3, -x_4, x_1, x_2)$$
$$\text{and} \quad f_3(x_1, x_2, x_3, x_4) := (-x_4, x_3, -x_2, x_1)$$

*Remark.* We can identify $\mathbb{S}^3$ with the unit sphere in the Quaternions $\mathbb{H}$. Then $f_1 = ix$, $f_2 = jx$ and $f_3 = kx$.

**(c)** Let $x_N := (1, 0, 0) \in \mathbb{S}^2$ and $x_S := (-1, 0, 0) \in \mathbb{S}^2$. With the aid of stereographic projection $\phi_N$ and $\phi_S$, write down local trivialisations of $T\mathbb{S}^2$ over $U_N := \mathbb{S}^2 \setminus \{x_N\}$ and $U_S := \mathbb{S}^2 \setminus \{x_S\}$, and calculate the transition function $g_{U_N,U_S} : \mathbb{S}^2 \setminus \{x_N, x_S\} \to \mathrm{GL}(\mathbb{R}^2)$. *(8 Points)*

*Remark.* $T\mathbb{S}^2$ is not trivial, but this require some more theory to prove. It is a consequence of the "hairy ball theorem": every global section of $T\mathbb{S}^2$ has a zero.

## Additional Exercises

**38. Sections of vector bundles.**

Let $(E, B, \pi)$ be a $\mathbb{R}$-vector bundle with fibre $F$, $s, s_1, s_2 : B \to E$ smooth sections of $(E, B, \pi)$, and $f : B \to \mathbb{R}$ a smooth function. Prove:

**(a)** The *zero section* $O : B \to E$, $b \mapsto 0_b$ is a smooth section. By $0_b$ we mean this: $F_b = \pi^{-1}[\{b\}]$ is a vector space, so it has a zero element $0_b \in F_b \subset E$. *(2 Points)*

**(b)** $s_1 + s_2$ and $g \cdot s$ are smooth sections. *(2 Points)*

**(c)** Interpret $f$ as a global section of the trivial bundle $E = \mathbb{R} \times B$. *(1 Point)*

**(d)** The image $s[B]$ is a submanifold of $E$. *(3 Points)*

**39. The tangent bundle of a product.**

Why is $T(X \times Y) = TX \times TY$. Consult Definition 1.41 and try to write $T_{(x,y)}(X \times Y)$ as a product.

**40. Line bundles over $\mathbb{R}$ are trivial.** Prove that every line bundle (a vector bundle whose fibre dimension is 1) over $\mathbb{R}$ is trivial. *(8 Points)*

Hint. Let $(E, \mathbb{R}, \pi)$ be a line bundle. Choose a point $x_0$ and show that there is an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ with a non-vanishing section $s$. Then consider

$$J := \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{There exists an extension } s_x \text{ of } s \text{ to } (x, x_0 + \varepsilon) \text{ or } (x_0 - \varepsilon, x), \\ \text{such that } s_x \text{ is non-vanishing} \end{array} \right\},$$

where the choice of $(x, x_0 + \varepsilon)$ or $(x_0 - \varepsilon, x)$ depends whether $x \leq x_0$ or $x \geq x_0$ Show that $J$ is non-empty, open. Argue further that $J = \mathbb{R}$.

**Terminology**

Schnitt = section

nullstellenfreien = non-vanishing

Geradenbündel = line bundle

American spelling is fiber, British spelling is fibre.