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Analysis III
5. Exercise: Submanifolds

10 October 2022

Preparation Exercises

26. Immersions between Euclidean Space.

A map is called immersive at a point if the tangent map is injective. This is the case if it is *full-rank*. The rank is the number of linearly independent columns. From the discussion following Definition 1.33 in the script and in last weeks exercises, between subsets of $\mathbb{R}^n$ the tangent map is just the Jacobian (first derivative matrix).

At which points are the following maps immersive?

(a) $f : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto f(t) = (\cos(2t), \sin(2t), t)$.

(b) $g : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto g(t) = (t^2, t^3)$. Is g injective?

(c) $h : (1, \infty) \to \mathbb{R}^2,\ t \mapsto h(t) = \left(\frac{t+1}{2t}\cos(2t), \frac{t+1}{2t}\sin(2t)\right)$.

Solution.

(a)

$$T_t(f) = (-2\sin(2t), 2\cos(2t), 1).$$

The last column is never zero, so the rank is 1. Thus f is an immersion.

(b)

$$T_t(g) = (2t, 3t^2)$$

This has rank 1, except where both columns vanish simultaneously, namely $t = 0$. This map is an injection however, since $t^3 = s^3 \Rightarrow t = s$.

(c)

$$T_t(h) = \left(-\frac{1}{2t^2}\cos(2t) - \frac{1+t}{t}\sin(2t),\ -\frac{1}{2t^2}\sin(2t) + \frac{1+t}{t}\cos(2t)\right).$$

This is rank 1, except if both columns vanish simultaneously. That occurs when

$$\begin{aligned} 0 &= \left(\cos(2t) + 2t(1+t)\sin(2t)\right)^2 + \left(\sin(2t) - 2t(1+t)\cos(2t)\right)^2 \\ &= 1 + 4t^2(1+t)^2. \end{aligned}$$

We see that in fact the two columns are never simultaneously zero, so h is an immersion.

27. Alternative version of the Constant Rank Theorem.

Let M and N be manifolds of dimensions m and n respectively, and $f : M \to N$ a map of constant rank r. Consider the standard set-up: at every point $p \in X$ there exists charts $\phi : U \to \mathbb{R}^m$ of M and $\psi : V \to \mathbb{R}^n$ of N with $p \in U$, $\phi(p) = 0$, $f[U] \subset V$ and $\psi(f(p)) = 0$.

Show: There exists such charts ϕ and ψ with the further property that $\psi \circ f \circ \phi^{-1}$ has the form

$$\psi \circ f \circ \phi^{-1} : \phi[U] \to \mathbb{R}^n,\ (y_1, \dots, y_m) \mapsto (y_1, \dots, y_r, \underbrace{0, \dots, 0}_{n-r}) . \qquad \textit{(3 Points)}$$

[Hint. Apply Theorem 1.44.]

Remark. This theorem shows that maps of constant rank can be written as a composition of a submersion and an immersion in a neighbourhood of every point (i.e. locally). Can you see the connection between this result and the first isomorphism theorem of linear algebra?

Solution. We apply Theorem 1.44 as suggested. Thus we know that $\psi \circ f \circ \phi^{-1}$ is equal to some linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ restricted to $\phi[U]$. The rest is just linear algebra.

We know that there is a basis of $\mathbb{R}^m$ such that A has row eschleon form. Further choose the basis $\{Ae_j | 1 \leq j \leq r\}$ for the image of A and extend it to a basis of $\mathbb{R}^n$. In these bases A has the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} .$$

Let L and R be the change of base matrices of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. We can view these as invertible linear operators $\mathbb{R}^m \to \mathbb{R}^m$ and $\mathbb{R}^n \to \mathbb{R}^n$. Finally, notice that $L^{-1} \circ \phi$ is a chart of M and $\mathbb{R} \circ \psi$ is a chart of N, and in these charts f has the desired expression in local coordinates.

Clearly a map which has this form has locally constant rank. Hence the result in this exercise is equivalent to the constant rank theorem. In fact, many books call this the constant rank theorem.

In Class Exercises

28. Submanifolds

(a) Thus far in the course we have defined manifold structure on $\mathbb{S}^1$ by giving an atlas (stereographic projection). We have also encountered the inclusion map ι which sends $\mathbb{S}^1$ as an abstract manifold to its subset of $\mathbb{R}^2$. Show that ι is an embedding.

(b) Consider Example 1.18(iv) from the lecture script. It says: Let $M = f^{-1}[\{0\}]$ be the preimage of 0 of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ whose gradient ∇f has no common zeroes with f. Then M is a manifold.
Show that the inclusion map is an embedding.

(c) What is the connection between the previous exercise and the constant rank theorem (Theorem 1.44/Corollary 1.46)?

Solution.

(a) To show a map is an embedding, you must show that it is an immersion and a homeomorphism onto its image.

Clearly ι is a bijection onto its image. We defined $\mathbb{S}^1$ to have the subspace topology, so ι is automatically a homeomorphism.

Since $\mathbb{S}^n$ is compact, it remains to show that it is an immersion. We will use the coordinate projection charts. Choose any point $x \in S^1$. Without loss of generality, assume that $x_1 > 0$. Let $h(y) = \sqrt{1-y^2}$. Then the tangent map in local coordinates is

$$J(\mathrm{id} \circ \iota \circ \Pi_1^{-1}) = \begin{pmatrix} \frac{\partial h}{\partial y} \\ 1 \end{pmatrix}$$

which is clearly rank 1. Hence this is an immersion.

(b) This is really a generalisation of Exercise 9, because the coordinate projection charts are exactly the charts one gets by using the implicit function theorem to construct an atlas on a level set. Let's give the detail though. M is a closed subset of $\mathbb{R}^{n+1}$ so it is Hausdorff and Lindelöf. We now give it charts. Take any point $p \in M$. By the assumption that the gradient does not vanish, there is a coordinate x_k such that $\partial_k f(p) \neq 0$. Without loss of generality, assume $k = 0$. The implicit function theorem says that there is a smooth height function h such that $\tilde{h} : y \mapsto (h(y), y)$ is an inverse to $\phi := \Pi_0|_{M \cap U}$ for some neighbourhoods U of $p \in M \subset \mathbb{R}^{n+1}$ and $\Pi_0[U]$ of $\Pi_0(p) \in \mathbb{R}^n$, where $\Pi_0(x_0, \ldots, x_n) = (x_1, \ldots, x_n)$ is the coordinate projection. Π_0 is a continuous function on $\mathbb{R}^{n+1}$, so if M is given the subspace topology, then the restriction of Π_0 to M is also continuous. The function $\tilde{h} : \Pi_0[U] \to \mathbb{R}^{n+1}$ has continuous components, and so is also continuous. Thus ϕ is a chart of M near p. We have previously shown that different coordinate projections are compatible in the sense of charts (essentially Exercise 11). So this makes M a manifold.

Let $\iota = \mathrm{id}|_M : M \to \mathbb{R}^{n+1}$ be the inclusion map, which is the identity map restricted to M but considered as a map between two manifolds. Clearly it is a bijection between M and $\iota[M]$. As in the previous part, ι is a homeomorphism because we gave M the subspace topology.

Here is another way to see that ι^{-1} is continuous that shows an useful tactic. Π_0 is a continuous function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Then $\tilde{h}$ is a continuous from a subset of $\mathbb{R}^n$ to a neighbourhood in M. The final piece is that $\iota^{-1}|_{M\cap U} = \tilde{h} \circ \Pi_0|_{M\cap U}$, which shows ι^{-1} locally as the composition of two continuous functions.

It remains to show that ι is an immersion. This is clear from the form of the inverse of $\phi = \Pi_0|_{M\cap U}$ given by the implicit function theorem:

$$J(\mathrm{id} \circ \iota \circ \phi^{-1}) = J\tilde{h} = \begin{pmatrix} \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} & \cdots & \frac{\partial h}{\partial y_n} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

(c) The result in Example 1.18(iv) is a special case of the constant rank theorem (or Corollary 1.46): $\nabla f \neq 0$ implies that the rank is always 1 so the preimage of a point is a submanifold of $\mathbb{R}^{n+1}$.

29. An inclusion map that is not an embedding.

Consider the subset $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^2 = x_1^3\}$ of the plane. This is the cusp curve. As a topology space (using the subspace topology) it is homeomorphic to the real line. Let

$$\phi : C \to \mathbb{R}, \quad (x_1, x_2) \mapsto \begin{cases} x_2/x_1 & \text{for } x_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The inverse of this map is

$$\phi^{-1}(y) = (y^2, y^3).$$

(a) Prove that $\{\phi\}$ is an atlas for C.

(b) Show that the inclusion map is not an embedding. This shows that $(C, \{\phi\})$ is 'not compatible' with the smooth structure of $\mathbb{R}^2$ in some sense.

(c) Does this show that C is not a submanifold.

Solution.

(a) ϕ is bijective because we gave its inverse. ϕ^{-1} is clearly continuous. ϕ is continuous for $x_1 \neq 0$ because it is the restriction of a continuous function there. At $(0, 0)$

$$\lim_{x_1\to 0} \phi(x_1, x_2) = \lim_{x_1\to 0} \sqrt{\left(\frac{x_2}{x_1}\right)^2} = \lim_{x_1\to 0} \sqrt{x_1} = 0 = \phi(0, 0).$$

This shows ϕ is continuous at this point.

(b) ι is a homeomorphism, because we used the subspace topology. So if it is not an embedding, it must not be an immersion.

Indeed it is not an immersion. Consider the curve $\alpha_1(t) = \phi^{-1}(t) = (t^2, t^3)$. This is a non-zero vector in $T_{(0,0)}C$. But

$$T_{(0,0)}(\iota)([\alpha]) = J_0(\iota \circ \alpha) = (2t, 3t^2)\Big|_{t=0} = (0,0).$$

This shows that $T_{(0,0)}(\iota)$ is not injective.

(c) This only proves that this particular map ι is not an embedding. To show that C is not a submanifold we need to show there does not exist any homeomorphic immersion $f : X \to C$ for any manifold X.

30. Embedding $\mathbb{R}/\mathbb{Z}$ into $\mathbb{R}^2$.

Consider the map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ given by $f([t]) = (\cos 2\pi t, \sin 2\pi t)$. Show that this is an embedding.

Solution. Similar to Exercise 17, this is a well-defined smooth map. The image is clearly the subset $\{x_0^1 + x_1^2 = 1\}$.

First note that f is a injective. If $(\cos 2\pi t, \sin 2\pi t) = (\cos 2\pi t', \sin 2\pi t')$ then $t' = t + n$ for some integer n. But then $[t] = [t']$ in $\mathbb{R}/\mathbb{Z}$.

Next we show it is an immersion. Choose any point $[x] \in \mathbb{R}/\mathbb{Z}$. In coordinates, f is

$$\mathrm{id} \circ f \circ \phi_x^{-1}(t) = (\cos 2\pi t, \sin 2\pi t),$$

a function from $(x - 0.5, x + 0.5) \subset \mathbb{R} \to \mathbb{R}^2$. Its Jacobian at x is

$$2\pi \begin{pmatrix} -\sin 2\pi x \\ \cos 2\pi x \end{pmatrix}.$$

From the relation $(-\sin 2\pi x)^2 + (\cos 2\pi x)^2 = 1$, it is impossible that both components are zero at the same time. Therefore f has rank 1 everywhere.

Finally, we must show that f is a homeomorphism. We know that it is a smooth map, so it is continuous.

To show that f^{-1} is continuous is quite difficult. Choose a point $(x_0, x_1) \in \mathbb{S}^1$. Suppose that $x_0 > 0$, the other cases will be similar. $(x_0, x_1) \mapsto \arctan(x_1/x_0)$ is a continuous map on $\{x_0 > 0\}$ and writing

$$f^{-1}|_{x_0>0}(x_0, x_1) = p\left(\frac{1}{2\pi}\arctan(x_1/x_0)\right)$$

shows that f^{-1} is continuous at these points.

31. The tangent space of submanifolds.

Let X , Y be manifolds and $f : X \to Y$ be a smooth map with constant rank. Then we know that for every $y \in f[X]$ the preimage $M := f^{-1}[\{y\}]$ is a submanifold of X. Show the following holds for $x \in M$:

$$T_x M = \ker T_x(f) \ .$$

Hint. Take $[\alpha] \in T_x M$, so a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ through x. Then consider the curve $f \circ \alpha$ in Y.

Remark. This is the 'complement' of the idea that for an embedding $\iota : M \to X$ the tangent vectors to $i[M]$ considered as a subset of X are $\operatorname{img} T_x(\iota) \subset T_x X$

Solution. This is one part of Corollary 1.46, but we give a more explicit proof.

Following the hint, take a vector in the submanifold $[\alpha] \in T_x M$. Then we have $f \circ \alpha(t) = y$ for all t, because the submanifold is a level set of f. Thus $T_x(f)([\alpha]) = [f \circ \alpha] = 0 \in T_{f(x)} Y$ because the constant map of a point represents the zero tangent vector. This shows that $[\alpha]$ belongs to the kernel of $T_x(f)$. Therefore $T_x M \subseteq \ker T_x(f)$.

On the other hand, we have seen in exercise with the alternative version of the constant rank theorem that there are charts ϕ on X and ψ on Y so that

$$\psi \circ f \circ \phi^{-1}(x_1, \dots, x_{\dim X}) = (x_1, \dots, x_r, 0, \dots) \in \mathbb{R}^r \times \mathbb{R}^{\dim Y - r}.$$

This shows that M has dimension $\dim X - r$ and so must $T_x M$. But by the rank-nullity theorem of linear algebra, $\dim T_x X = \dim \ker T_x(f) + r$, which shows that $\ker T_x(f)$ has the same dimension as $T_x M$. If a subspace has the same dimension as the space it lies in, it must be the whole space. Hence $T_x M = \ker T_x(f)$.

Additional Exercises

32. More examples of submanifolds.

(a) Let $a > 0$ and $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = \left(a - \sqrt{x^2 + y^2}\right)^2 + z^2.$$

Show that the preimage $f^{-1}[\{b^2\}]$ is a submanifold of $\mathbb{R}^3$ for every b with $0 < b < a$. What is this space?

(b) Investigate for which values $t \in \mathbb{R}$

$$A_t := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = t \}$$

is a submanifold of $\mathbb{R}^{n+1}$. What is this space?

(4 Points)

Solution. If the gradient does not vanish on the preimage, then Example 1.18(iv) shows that they are manifolds and the previous exercise shows that they are submanifolds. In (a), the gradient vanishes only at the origin, which does not belong to the preimage because $b < a$. In (b) the gradient also only vanishes at the origin, and this belongs to the level set A_0. The spaces in (a) are tori (doughnut/donut shape), and in (b) for $t > 0$ it is one-sheeted hyperboloid and for $t < 0$ it is a two-sheet hyperboloid.

33. The Cartesian product of manifolds.

Let M and N be manifolds with dimensions m and n respectively. Verify the claim from Section 1.6 that "the map $\phi \times \psi$ is a chart of $M \times N$ and the collection of all such maps is an atlas." This shows that $M \times N$ is a manifold with dimension $m + n$.

You may assume that the map is homeomorphism if you don't want to try to understand the topology of the product. In that case, you must still show each map is bijective and they are compatible. *(4 Points)*

Solution. First, let us recall what we mean by $P := M \times N$. It is the set of pairs (x, y) with $x \in M$ and $y \in N$. This set can be given a topology: $S \subset P$ is open if and only if for every point $(x, y) \in P$ there are neighbourhoods $U \subset M$ of x and $V \subset N$ of y such that $U \times V \subset S$. This is called the *product topology*.

This way of defining a topology is similar to using balls to define the topology of $\mathbb{R}^n$; a ball is a special type of neighbourhood of $\mathbb{R}^n$. Indeed, if we consider $\mathbb{R} \times \mathbb{R}$ as an example, then the special neighbourhoods are open rectangles, or balls in the 1-norm. Since we know from Analysis II that every norm on $\mathbb{R}^n$ gives the same topology, the product topology of $\mathbb{R}^2$ is the same as its usual topology.

The lecture notes address the question of whether the product is Hausdorff and Lindelöf: it is. So we proceed to examine the charts.

Let $\phi : U \to \mathbb{R}^m$ be a chart of M and $\psi : V \to \mathbb{R}^n$ be a chart of N. We claim that

$$\phi \times \psi : U \times V \to \mathbb{R}^{m+n}, \quad \phi \times \psi(x, y) := (\phi(x), \psi(y))$$

is a chart of P. It is a bijection because $\phi^{-1} \times \psi^{-1}(v, w) := (\phi^{-1}(v), \psi^{-1}(w))$ is its inverse. Let $f = \phi \times \psi$.

You might want to skip this step: We next show that the function is a homeomorphism. Note that $U \times V$ is an open set of P by definition of the product topology. Choose any open set $S \subset f(U \times V) \subset \mathbb{R}^{m+n}$ and consider $R := f^{-1}[S]$. Choose any point of (x, y) of R. Because S is open, there is an open rectangle $I \times J \subset S$ containing $f(x, y)$. Then $f^{-1}[I \times J] = \phi^{-1}[I] \times \psi^{-1}[J]$ is an open neighbourhood of (x, y) since ϕ and ψ are continuous. Since this holds for every point of S, this set is open. The proof that f^{-1} is continuous is the same.

Hence f is a chart of P. Finally, we must show that all such charts are compatible. But consider the transition functions, they have the form:

$$\tilde{f} \circ f^{-1}(v, w) = (\tilde{\phi} \times \tilde{\psi})(\phi^{-1}(v), \psi^{-1}(w)) = (\tilde{\phi}(\phi^{-1}(v)), \tilde{\psi}(\psi^{-1}(w))).$$

The components of this map are just transitions function of M and N, which are smooth. And since these are maps between open subsets of Euclidean space, they are smooth if and only if the components are smooth. Therefore all the transition functions are smooth, i.e. the charts are all compatible.

34. Immersion of compact manifolds.

Let M be an n-dimensional compact manifold and $f : M \to \mathbb{R}^n$ a smooth map. Show that f cannot be an immersion.

Hint. Investigate the topological properties of $f[M]$.

Solution. Immediately we can say that $N = f[M]$ is a compact subset of $\mathbb{R}^n$, therefore closed. Suppose that f is an immersion. That means that $T(f)$ is rank n at every point. But then f is also a submersion. This makes f a local diffeomorphism and so the image N must also be open. But the only closed and open sets in $\mathbb{R}^n$ are the empty set and $\mathbb{R}^n$ itself. N cannot be empty and $\mathbb{R}^n$ is not compact. Therefore we have a contradiction: f cannot be an immersion.

Can you generalise this result for $f : M \to \mathbb{R}^m$, or provide counter-examples?

Terminology

Umkehrsatz = Inverse Function Theorem.
Rangsatz = Constant Rank Theorem.
Rang = rank (symbol is rk).