Martin Schmidt
Ross Ogilvie

Analysis III
5. Exercise: Submanifolds

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Preparation Exercises

26. Immersions between Euclidean Space.

A map is called immersive at a point if the tangent map is injective. This is the case if it is *full-rank*. The rank is the number of linearly independent columns. From the discussion following Definition 1.33 in the script and in last weeks exercises, between subsets of $\mathbb{R}^n$ the tangent map is just the Jacobian (first derivative matrix).

At which points are the following maps immersive?

(a) $f : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto f(t) = (\cos(2t), \sin(2t), t)$.

(b) $g : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto g(t) = (t^2, t^3)$. Is g injective?

(c) $h : (1, \infty) \to \mathbb{R}^2,\ t \mapsto h(t) = \left(\frac{t+1}{2t}\cos(2t), \frac{t+1}{2t}\sin(2t)\right)$.

27. Alternative version of the Constant Rank Theorem.

Let M and N be manifolds of dimensions m and n respectively, and $f : M \to N$ a map of constant rank r. Consider the standard set-up: at every point $p \in X$ there exists charts $\phi : U \to \mathbb{R}^m$ of M and $\psi : V \to \mathbb{R}^n$ of N with $p \in U$, $\phi(p) = 0$, $f[U] \subset V$ and $\psi(f(p)) = 0$.

Show: There exists such charts ϕ and ψ with the further property that $\psi \circ f \circ \phi^{-1}$ has the form

$$\psi \circ f \circ \phi^{-1} : \phi[U] \to \mathbb{R}^n,\ (y_1, \ldots, y_m) \mapsto (y_1, \ldots, y_r, \underbrace{0, \ldots, 0}_{n-r}) . \qquad \textit{(3 Points)}$$

[Hint. Apply Theorem 1.44.]

Remark. This theorem shows that maps of constant rank can be written as a composition of a submersion and an immersion in a neighbourhood of every point (i.e. locally). Can you see the connection between this result and the first isomorphism theorem of linear algebra?

In Class Exercises

28. Submanifolds

(a) Thus far in the course we have defined manifold structure on $\mathbb{S}^1$ by giving an atlas (stereographic projection). We have also encountered the inclusion map ι which sends $\mathbb{S}^1$ as an abstract manifold to its subset of $\mathbb{R}^2$. Show that ι is an embedding.

(b) Consider Example 1.18(iv) from the lecture script. It says: Let $M = f^{-1}[\{0\}]$ be the preimage of 0 of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ whose gradient ∇f has no common zeroes with f. Then M is a manifold.
Show that the inclusion map is an embedding.

(c) What is the connection between the previous exercise and the constant rank theorem (Theorem 1.44/Corollary 1.46)?

29. An inclusion map that is not an embedding.

Consider the subset $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^2 = x_1^3\}$ of the plane. This is the cusp curve. As a topology space (using the subspace topology) it is homeomorphic to the real line. Let

$$\phi : C \to \mathbb{R}, \quad (x_1, x_2) \mapsto \begin{cases} x_2/x_1 & \text{for } x_1 \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

The inverse of this map is

$$\phi^{-1}(y) = (y^2, y^3).$$

(a) Prove that $\{\phi\}$ is an atlas for C.

(b) Show that the inclusion map is not an embedding. This shows that $(C, \{\phi\})$ is 'not compatible' with the smooth structure of $\mathbb{R}^2$ in some sense.

(c) Does this show that C is not a submanifold.

30. Embedding $\mathbb{R}/\mathbb{Z}$ into $\mathbb{R}^2$.

Consider the map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ given by $f([t]) = (\cos 2\pi t, \sin 2\pi t)$. Show that this is an embedding.

31. The tangent space of submanifolds.

Let X , Y be manifolds and $f : X \to Y$ be a smooth map with constant rank. Then we know that for every $y \in f[X]$ the preimage $M := f^{-1}[\{y\}]$ is a submanifold of X. Show the following holds for $x \in M$:

$$T_x M = \ker T_x(f) \ .$$

Hint. Take $[\alpha] \in T_x M$, so a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ through x. Then consider the curve $f \circ \alpha$ in Y.

Remark. This is the 'complement' of the idea that for an embedding $\iota : M \to X$ the tangent vectors to $i[M]$ considered as a subset of X are $\operatorname{img} T_x(\iota) \subset T_x X$

Additional Exercises

32. More examples of submanifolds.

(a) Let $a > 0$ and $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = \left(a - \sqrt{x^2 + y^2}\right)^2 + z^2.$$

Show that the preimage $f^{-1}[\{b^2\}]$ is a submanifold of $\mathbb{R}^3$ for every b with $0 < b < a$. What is this space?

(b) Investigate for which values $t \in \mathbb{R}$

$$A_t := \{\, (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = t \,\}$$

is a submanifold of $\mathbb{R}^{n+1}$. What is this space?

(4 Points)

33. The Cartesian product of manifolds.

Let M and N be manifolds with dimensions m and n respectively. Verify the claim from Section 1.6 that "the map $\phi \times \psi$ is a chart of $M \times N$ and the collection of all such maps is an atlas." This shows that $M \times N$ is a manifold with dimension $m + n$.

You may assume that the map is homeomorphism if you don't want to try to understand the topology of the product. In that case, you must still show each map is bijective and they are compatible. *(4 Points)*

34. Immersion of compact manifolds.

Let M be an n-dimensional compact manifold and $f : M \to \mathbb{R}^n$ a smooth map. Show that f cannot be an immersion.

Hint. Investigate the topological properties of $f[M]$.

Terminology

Umkehrsatz = Inverse Function Theorem.
Rangsatz = Constant Rank Theorem.
Rang = rank (symbol is rk).