Martin Schmidt
Ross Ogilvie

Analysis III
4. Exercise: Tangent Space

3 October 2022

In Analysis II we learn that the derivative of a function $F = (F_1, \dots, F_n) : \mathbb{R}^m \to \mathbb{R}^n$ at a point a is a linear map, an $n \times m$ matrix,

$$J_a F := \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a) & \dots & \frac{\partial F_1}{\partial x_m}(a) \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(a) & \dots & \frac{\partial F_n}{\partial x_m}(a) \end{pmatrix}.$$

I call this matrix the Jacobian and denote it with the letter J. This is so we do not confuse it with derivations, which often use the letter D. Martin uses a prime $F'(a)$. All three $J, D, '$ are common. When the matrix is a single column we identify it with the vector. ∇ is also a common notation in this case. When it is 1×1 we identify it with the real number. We have the chain rule

$$J_a(F \circ G) = J_{G(a)}F \circ J_a G.$$

When thinking of these are matrices and vectors, we may omit the $\circ$ and use juxtaposition to represent matrix multiplication.

Preparation Exercises

20. Tangency for curves.

In Definition 1.32 we define the concept of tangency for smooth maps f_1, f_2 at a point $x = f_1(t) = f_2(t)$. In this exercise we examine this concept for curves. A curve through $x \in X$ is a smooth map $\alpha : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to X$ such that $\alpha(0) = x$. Another way to say Definition 1.33 is that the tangent space $T_x X$ is the set of curves through x with the equivalence relation of tangency.

(a) Give the definition for two curves α, β through x to be tangential at x.

Let ϕ be a chart that contains x. Show that these curves are tangential at x if and only if $J_0(\phi \circ \alpha) = J_0(\phi \circ \beta) \in \mathbb{R}^n$.

(b) Choose $w \in \mathbb{R}^n$. Show that $\alpha_w(t) := \phi^{-1}(tw + \phi(x))$ is a curve through x.

(c) Show that every curve through x is equivalent to α_w for some w. This shows that $T_x X = \{[\alpha_w] \mid w \in \mathbb{R}^n\}$. This correspondence is called 'writing a tangent vector in coordinates'.

Solution.

(a) In terms of Definition 1.32, we must compare α and β in the charts id on $\mathbb{R}$ and ϕ on X. That is, we must compare the derivatives of $\phi \circ \alpha \circ \mathrm{id}^{-1} = \phi \circ \alpha$ and $\phi \circ \beta$ at 0 as linear maps $\mathbb{R} \to \mathbb{R}^n$. In our situation, the Jacobian is a single column $J_0(\phi \circ \alpha) = \frac{d}{dt}(\phi \circ \alpha)\big|_{t=0}$. Because it is only a single column, we identify this matrix with a vector in $\mathbb{R}^n$. The two matrices are the same if and only if the two vectors are the same.

(b) We have seen previously that every coordinate chart is a diffeomorphism. We recognise then that α_v is the composition of the smooth map $(-\varepsilon, \varepsilon) \to \phi[U] \subset \mathbb{R}^n, t \mapsto tw + \phi(x)$ and ϕ^{-1}, where we choose ε small so that it lies in $\phi[U]$. This shows it is a smooth map $(-\varepsilon, \varepsilon) \to X$. And $\alpha_w(0) = \phi^{-1}(\phi(x)) = x$.

(c) We see directly that $\phi \circ \alpha_w(t) = \phi \circ \phi^{-1}(tw + \phi(x)) = tw + \phi(x)$ so $J_0(\phi \circ \alpha_w) = w$. Choose any curve α through x and set $w = J_0(\phi \circ \alpha)$. Then by part (a), α and α_w are tangent at x.

21. Derivations at a point.

We refer to Theorem 1.40 in the script and the explanation that proceeds it.

(a) Consider $D : C^\infty(X, \mathbb{R}) \to \mathbb{R}$ and a point $x \in X$. Recall the definition that D is a derivation at x.

(b) Prove that the set of derivations at x is a vector space.

(c) Let α be a curve through x. Show that $D_\alpha : C^\infty(X, \mathbb{R}) \to \mathbb{R}$ given by $D_\alpha(f) = (f \circ \alpha)'(0)$ is a derivation at x.

(d) Show that evaluating a partial derivative of a function with respect to a chart ϕ (see Exercise 14(e)) is a derivation at x. That is,

$$f \mapsto \frac{\partial f}{\partial \phi_i}(x) = \frac{\partial (f \circ \phi^{-1})}{\partial y_i}(\phi(x)).$$

Solution.

(a) D should be $\mathbb{R}$-linear. This means for all $f, g \in C^\infty(X, \mathbb{R})$ and $a, b \in \mathbb{R}$ we have $D(af + bg) = aD(f) + bD(g)$. Additionally, it should satisfy Leibniz rule at x, ie $D(fg) = f(x)D(g) + g(x)D(f)$. Note how this depends on the point x.

(b) Suppose D, D' are derivations at x and $a, b \in \mathbb{R}$. Then $aD + bD'$ is a derivation. The $\mathbb{R}$-linearity is standard (but a good exercise if you are a little rusty) so I won't write it out but I'll check the Leibniz rule:

$$\begin{aligned}(aD + bD')(fg) &= aD(fg) + bD'(fg) \\ &= a[f(x)D(g) + g(x)D(f)] + b[f(x)D'(g) + g(x)D'(f)] \\ &= f(x)[aD(g) + bD'(g)] + g(x)[aD(f) + bD'(f)] \\ &= f(x)[aD + bD'](g) + g(x)[aD + bD'](f).\end{aligned}$$

The zero element of this vector space is the derivation D_0 that sends every function to zero, ie $D_0(f) = 0$. This is sufficient to show it is a vector space.

(c) The important thing to recognise is that $f \circ \alpha$ is a function from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$, nothing scary. I could have written it as J_0, but I wanted to emphasis this is the completely basic derivative of a real function. The Leibniz rule follows from the normal Leibniz for the product of real functions (which we don't have for vector valued functions). Observe

$$\begin{aligned} D_\alpha(af + bg) &= ((af + bg) \circ \alpha)'(0) = (a(f \circ \alpha) + b(g \circ \alpha))'(0) \\ &= a(f \circ \alpha)'(0) + b(g \circ \alpha)'(0) = aD_\alpha(f) + bD_\alpha(g), \end{aligned}$$

and

$$\begin{aligned} D_\alpha(fg) &= ((fg) \circ \alpha)'(0) = ((f \circ \alpha)(g \circ \alpha))'(0) \\ &= (f \circ \alpha)(0)(g \circ \alpha)'(0) + (f \circ \alpha)'(0)(g \circ \alpha)(0) \\ &= f(x)D_\alpha(g) + D_\alpha(f)g(x). \end{aligned}$$

By the way, another way to write this derivation is $D_\alpha(f) = T_x(f)(\alpha)$ using the identification $T_{f(x)}\mathbb{R} = \mathbb{R}$.

(d) This is very similar to the previous question (can you give a reason for that?). I'll just check the Leibniz rule. Perhaps if we write $\tilde{f} = f \circ \phi^{-1} : \phi[U] \subset \mathbb{R}^n \to \mathbb{R}$ this will be clearer:

$$\begin{aligned} \frac{\partial(fg)}{\partial\phi_i}(x) &= \frac{\partial((fg) \circ \phi^{-1})}{\partial y_i}(\phi(x)) = \frac{\partial((f \circ \phi^{-1})(g \circ \phi^{-1}))}{\partial y_i}(\phi(x)) = \frac{\partial(\tilde{f}\tilde{g})}{\partial y_i}(\phi(x)) \\ &= \tilde{f}(\phi(x))\frac{\partial\tilde{g}}{\partial y_i}(\phi(x)) + \tilde{g}(\phi(x))\frac{\partial\tilde{g}}{\partial y_i}(\phi(x)) \\ &= f(x)\frac{\partial g}{\partial\phi_i}(x) + g(x)\frac{\partial g}{\partial\phi_i}(x). \end{aligned}$$

In Class Exercises

22. The tangent map.

Let X, Y be manifolds and $f : X \to Y$. The map $T_x(f) : T_xX \to T_{f(x)}Y$ is called the tangent map is defined in Definition 1.35. It is also called the push-forward map or the differential.

(a) Prove that if α and β are curves through x that are tangential, then $T_x(f)(\alpha)$ is tangential to $T_x(f)(\beta)$ at $f(x)$. This shows that $T_x(f)$ is indeed well-defined between tangent spaces.

(b) Suppose that $Y = \mathbb{R}^m$. Using the canonical identification $T_y\mathbb{R}^m = \mathbb{R}^m$ show how to identify $T_x(f)(\alpha)$ with a vector in $\mathbb{R}^m$. How does this relate to Exercise 20(a)?

(c) Let X be connected. Show that f is constant if and only if $T_x(f) = 0$ for all $x \in X$.

Solution.

(a) Let ϕ be a chart centered on x and ψ a chart containing $f(x)$. $T_x(f)(\alpha)$ is by definition the composition $f \circ \alpha$. Then

$$\begin{aligned}
J_0(\psi \circ T_x(f)(\alpha)) &= J_0(\psi \circ f \circ \alpha) \\
&= J_0((\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \alpha)) \\
&= J_0(\psi \circ f \circ \phi^{-1}) J_0(\phi \circ \alpha) \\
&= J_0(\psi \circ f \circ \phi^{-1}) J_0(\phi \circ \beta) \\
&= J_0(\psi \circ T_x(f)(\beta)).
\end{aligned}$$

(b) $T_x(f)(\alpha)$ is the vector $[f \circ \alpha] \in T_{f(x)}\mathbb{R}^m$. If we write this in coordinates using the chart id, we get

$$J_0(\mathrm{id}^{-1} \circ (f \circ \alpha)) = J_0(f \circ \alpha) = \begin{pmatrix} \left.\frac{\partial (f_1 \circ \alpha)}{\partial t}\right|_{t=0} \\ \vdots \\ \left.\frac{\partial (f_m \circ \alpha)}{\partial t}\right|_{t=0} \end{pmatrix}.$$

In particular, if we take $f = \phi : U \to \phi[U] \subset \mathbb{R}^n$ we get that a writing a tangent vector in coordinates is the same thing as applying $T_x(\phi)$.

(c) Suppose that f is constant. Then $f(x) = q$ for some $q \in Y$ and all $x \in X$. Choose any point $x \in X$ and a curve α through x. Then the push-forward $f \circ \alpha$ is the constant map $t \to q$. This is the zero element of the tangent space.

Conversely, suppose that $T_x(f) = 0$. Then its rank is everywhere 0. By Corollary 1.46, for every point $y \in T[X]$ the preimage $f^{-1}[\{y\}]$ is a submanifold of dimension $\dim X - 0 = \dim X$. Submanifolds of the same dimension must be open, and because it is the preimage of a point it is also closed. Because X is connected, the submanifold must therefore be all of X. In other words, $X = f^{-1}[\{y\}]$, so $f[X] = y$, which shows f is constant.

23. Examples of tangent vectors.

(a) Let $\alpha, \beta : \mathbb{R} \to \mathbb{S}^1$ be given by $\alpha(t) = (\sin t, \cos t)$ and $\beta(t) = (\sin t^2, \cos t^2)$. Do these curves through $(0, 1)$ give the same tangent vector in $T_{(0,1)}\mathbb{S}^1$?

(b) Write α in coordinates with respect to ϕ_N and ϕ_S.

(c) How do you transform a vector written in coordinates with respect to one chart into another chart?

(d) Let $\iota : \mathbb{S}^1 \to \mathbb{R}^2$ be the map $\iota(x_1, x_2) = (x_1, x_2)$. This is called the inclusion map. Let v be a tangent vector in $T_x\mathbb{S}^1$. Show that $w := T_x(\iota)(v) \in \mathbb{R}^2$ is perpendicular to x.

Conversely, choose any $w \in \mathbb{R}^2$ with $w \cdot x = 0$ and set $\alpha(t) = (\cos|w|t)x + (\sin|w|t)\widehat{w}$. Show that $w = T_x(\iota)([\alpha])$. *(2 Points)*

Hence we make the identification

$$T_x\mathbb{S}^1 = \{\, w \in \mathbb{R}^2 \mid w \cdot x = 0 \,\}\,.$$

(e) Recall the maps $f : \mathbb{S}^1 \to \mathbb{R}$ and $A : \mathbb{S}^1 \to \mathbb{S}^1$ from Exercise 16. Write $T_{(1,0)}(f)(\alpha)$ and $T_{(1,0)}(\iota \circ A)(\alpha)$ using natural identifications. Interpret these results.

Solution.

(a) We can use the test from Exercise 20(a) with the chart ϕ_S.

$$\begin{aligned}
(\phi_S \circ \alpha)(t) &= \phi_S(\sin t, \cos t) = \frac{\cos t}{1 + \sin t} \\
J_t(\phi_S \circ \alpha) &= \frac{-\sin t(1 + \sin t) - \cos^2 t}{(1 + \sin t)^2} = \frac{-1}{1 + \sin t} \\
J_0(\phi_S \circ \alpha) &= -1,
\end{aligned}$$

and

$$\begin{aligned}
(\phi_S \circ \beta)(t) &= \phi_S(\sin t^2, \cos t^2) = \frac{\cos t^2}{1 + \sin t^2} \\
J_t(\phi_S \circ \beta) &= \frac{-2t\sin t^2(1 + \sin t^2) - 2t\cos^2 t^2}{(1 + \sin t^2)^2} \\
J_0(\phi_S \circ \beta) &= 0.
\end{aligned}$$

Thus the curves are not tangential.

(b) We have already written α in coordinates with respect to ϕ_S in the previous part.

$$\begin{aligned}
(\phi_N \circ \alpha)(t) &= \phi_N(\sin t, \cos t) = \frac{\cos t}{1 - \sin t} \\
J_t(\phi_N \circ \alpha) &= \frac{-\sin t(1 - \sin t) + \cos^2 t}{(1 - \sin t)^2} = \frac{1}{1 - \sin t} \\
J_0(\phi_N \circ \alpha) &= 1.
\end{aligned}$$

We see that in different coordinates, the same vector $[\alpha]$ may have different representations.

(c) A vector written in coordinates with respect to ϕ_1 is a fancy name for $J_0(\phi_1 \circ \alpha)$.

$$J_0(\phi_2 \circ \alpha) = J_0(\phi_2 \circ \phi_1^{-1} \circ \phi_1 \circ \alpha) = J_{\phi_1(x)}(\phi_2 \circ \phi_1^{-1}) J_0(\phi_1 \circ \alpha)$$

by the chain rule. In other words, you change coordinate expressions of vectors, you multiply by the Jacobian matrix of the change of coordinates map.

(d) Let α be a curve in $\mathbb{S}^1$ representing v, that is $x = \alpha(0)$ and $v = [\alpha]$. Then $T_x(\iota)(v) = [\iota \circ \alpha]$. But ι is just the identity map considered as a map between manifolds so $\iota \circ \alpha$ is just $t \mapsto \alpha(t) \in \mathbb{R}^2$ and $w = \alpha'(0)$.

To show that w is perpendicular to x, note that $|\alpha(t)|^2 = 1$ because it lies in the circe. Differentiating gives $2\alpha(t) \cdot \alpha'(t) = 0$. At $t = 0$ this gives $x \cdot w = 0$.

Conversely, suppose $w \in \mathbb{R}^2$ is perpendicular to x. We need to find a path in $\mathbb{S}^1$ with this as its tangent vector: $\alpha(t) = (\cos|w|t)x + (\sin|w|t)\widehat{w}$ works.

This argument did not depend on the dimension, so the same result holds for $T_x\mathbb{S}^n$.

(e) If we just wanted to calculate tangent vectors, there is nothing to do here. $T_{(1,0)}(f)(\alpha)$ is by definition the equivalence class of $f \circ \alpha$ and $T_{(1,0)}(\iota \circ A)(\alpha)$ is the equivalence class of $\iota \circ A \circ \alpha$.

But let us explore a little deeper. We know that for $T_{(1,0)}(f)(\alpha)$ we can write this as a vector in $\mathbb{R}$ (ie a real number). We get the vector

$$J_0(f \circ \alpha) = \left.\frac{d}{dt}\right|_{t=0} f(\cos t, \sin t) = 2\cos t + 2\sin t \cos t\Big|_{t=0} = 2.$$

The interpretation is that as we move along α the function f is changing, and at $t = 0$ it is increasing with speed 2.

For $T_{(1,0)}(\iota \circ A)(\alpha)$ we get the curve $t \mapsto (-\cos t, -\sin t)$, a curve through $(-1, 0)$. Using the natural identification

$$\begin{aligned} T_{(1,0)}(\iota \circ A)(\alpha) &= \left.\frac{d}{dt}\right|_{t=0} (-\cos t, -\sin t) \\ &= (\sin t, -\cos t)\Big|_{t=0} \\ &= (0, -1). \end{aligned}$$

The antipodal map reflects the circle. It is not surprising therefore that it reflects tangent vectors.

24. The tangent space as a vector space.

We know that the tangent space T_xX is the set of equivalence classes of curves through x. We want this to be a vector space, but one cannot add curves to one another.

As in Theorem 1.36 let ϕ be a chart centered on x, $\phi(x) = 0$. Then because ϕ is a homeomorphism, the induced map $\Phi := T_x(\phi)$ is a bijection from $V := T_xX$ to $T_{\phi(x)}\mathbb{R}^n = \mathbb{R}^n$. We give V the structure of a vector space which makes Φ an isomorphism. Explicitly

$$[\alpha] + [\beta] = \Phi^{-1}\Big(\Phi([\alpha]) + \Phi([\beta])\Big),$$
$$a[\alpha] = \Phi^{-1}\Big(a\Phi([\alpha])\Big).$$

The question is, does the vector space structure depend on the choice of chart ϕ? If we add two vectors according to one chart, do we get the same answer to when we add them according to another chart?

Prove directly that the vector space structure on the tangent space does not depend on the choice of chart (Theorem 1.36(i)).

Solution. We begin by describing the inverse $\Phi^{-1} : \mathbb{R}^n \to V$ more carefully. Choose a $v \in \mathbb{R}^n$. We know from Exercise 20 that $\alpha_v = \phi^{-1}(tv)$ is a curve through x. $\Phi(\alpha_w)$ is defined to be the curve $\phi \circ \alpha_w \circ \mathrm{id}^{-1}(t) = tw$, which shows $\Phi^{-1}(w) = [\alpha_w]$.

Let ψ be another chart centered on x, and $\Psi = T_{\psi(x)}(\psi)$ and $\beta_w(t) = \Psi^{-1}(tw)$ be the construction in this chart. We need to show that $\alpha_{\Phi(\alpha)+\Phi(\beta)}$ and $\beta_{\Psi(\alpha)+\Psi(\beta)}$ are tangential at x and that $\alpha_{a\Phi(\alpha)}$ and $\beta_{a\Psi(\alpha)}$ are also tangential. We do this by comparing them in the ϕ chart:

$$J_0(\phi \circ \alpha_{\Phi(\alpha)+\Phi(\beta)}) = J_0\left(\phi \circ \phi^{-1}\Big(t[\Phi(\alpha) + \Phi(\beta)]\Big)\right) = \Phi(\alpha) + \Phi(\beta)$$

and

$$\begin{aligned}
J_0\left(\phi \circ \beta_{\Psi(\alpha)+\Psi(\beta)}\right) &= J_0\left(\phi \circ \psi^{-1}\Big(t[\Psi(\alpha) + \Psi(\beta)]\Big)\right)\\
&= J(\phi \circ \psi^{-1})_0\,[\Psi(\alpha) + \Psi(\beta)]\\
&= J(\phi \circ \psi^{-1})_0\,[J_0(\psi \circ \alpha) + J_0(\psi \circ \beta)]\\
&= J_0(\phi \circ \psi^{-1})J_0(\psi \circ \alpha) + J_0(\phi \circ \psi^{-1})J_0(\psi \circ \beta)\\
&= J_0(\phi \circ \psi^{-1} \circ \psi \circ \alpha) + J_0(\phi \circ \psi^{-1} \circ \psi \circ \beta)\\
&= J_0(\phi \circ \alpha) + J_0(\phi \circ \beta)\\
&= \Phi(\alpha) + \Phi(\beta),
\end{aligned}$$

where we used the chain rule $J(F \circ G) = JF \circ JG$ in the second line, and in reverse in the 5th line. Because these vectors are equal, by Exercise 20 we know that the curves are tangential. The proof for vector scaling is similar. This shows that the vector space structure does not depend on the choice of chart.

Additional Exercises

25. Tangent vectors and Derivations at a point.

Using the results already developed in this exercise sheet, give your own proof of Theorem 1.40. That is, prove that $[\alpha] \to D_\alpha$ is a well-defined bijection and that it preserves the vector space structure.

Terminology

f und g berühren = f and g are tangential.