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Analysis III
3. Exercise: Smooth Maps

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Preparation Exercises

14. Properties of Smooth Maps.

(a) Let $x \in X$ be a point that is in the domain of two charts $\phi_1 : U_1 \to \phi[U_1]$ and $\phi_2 : U_2 \to \phi[U_2]$. Let $f : X \to Y$ be a map. Show that whether f is smooth at x does not depend on the chart.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Prove that $g \circ f$ is a smooth map.

(c) Show that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth in the sense of Definition 1.22 (as a map between manifolds), exactly if it is smooth in the Euclidean sense.

(d) Let X be a manifold and $\phi : U \to \phi[U] \subset \mathbb{R}^n$ a chart. Explain why $\phi : U \to \mathbb{R}^n$ is a smooth map in the sense of manifolds.

(e) Let $f : X \to \mathbb{R}$ be a smooth function (we often use the word function for maps to $\mathbb{R}$, though they are interchangable). Choose a chart $\phi : U \to \phi[U] \subset \mathbb{R}^n$. Define for $1 \le i \le n$ the i^{th}-partial derivative of f with respect to ϕ as

$$\frac{\partial f}{\partial \phi_i} : U \to \mathbb{R}, \quad x \mapsto \frac{\partial (f \circ \phi^{-1})}{\partial y_i}(\phi(x)).$$

Show that this is a smooth function on U. It is sometimes written as $\partial f / \partial \phi_{,i}$ (especially in physics) to make clear that the index refers to the coordinate and not which chart from the atlas.

Solution.

(a) Without loss of generality we may assume $U_1 = U_2$ (by restricting to the intersection). Let ψ be a suitable chart on Y. Because

$$\psi \circ f \circ \phi_1^{-1} = (\psi \circ f \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$

and $\phi_2 \circ \phi_1^{-1}$ is smooth, if $\psi \circ f \circ \phi_2^{-1}$ is smooth then so too is $\psi \circ f \circ \phi_1^{-1}$. This also holds vice versa. Therefore it doesn't depend on the choice of chart.

By similar reasoning, it also doesn't depend on the choice of chart on Y.

(b) Let ϕ, ψ, χ be suitable charts on X, Y, Z respectively. Then

$$\chi \circ (g \circ f) \circ \phi^{-1} = (\chi \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})$$

shows that $g \circ f$ is smooth.

(c) The charts on Euclidean space are just the identity functions. So f is smooth in the sense of manifolds if and only if

$$\mathrm{id}_{\mathbb{R}^m} \circ f \circ \mathrm{id}_{\mathbb{R}^n}^{-1} = f$$

is smooth in the Euclidean sense.

(d) The trick here is to use ϕ as the map f and as the chart on X, while using id as the chart on $\phi[U] \subset \mathbb{R}^n$ (a previous exercise showed that an open subset of a manifold is itself a manifold with the same charts). Thus ϕ is smooth in the sense of manifolds if

$$\mathrm{id} \circ \phi \circ \phi^{-1} = \mathrm{id}$$

is smooth in the Euclidean sense. Clearly it is.

(e) Again, we should use ϕ as the chart on X and id as the chart on $\mathbb{R}$. The assumption that f is a smooth map between manifolds amounts to $f \circ \phi^{-1}$ being smooth in the Euclidean sense.

Notice that the formula we give for the partial derivative is itself a composition. Thus

$$\mathrm{id} \circ \frac{\partial f}{\partial \phi_i} \circ \phi^{-1} = \mathrm{id} \circ \left(\frac{\partial (f \circ \phi^{-1})}{\partial y_i} \circ \phi \right) \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial y_i} : \phi[U] \subset \mathbb{R}^n \to \mathbb{R}$$

This is the partial derivative of a smooth function in the Euclidean sense, so it a smooth function by definition.

15. Diffeomorphism.

Let X, Y be differential manifolds. Show that X and Y are diffeomorphic (Def 1.21) exactly when there is a bijective smooth map $f : X \to Y$ whose inverse is also smooth.

Solution. Suppose that there is a bijective smooth map $f : X \to Y$ whose inverse is also smooth. Then f is a homeomorphism. It remains to show that for any chart ψ of Y the composition $\psi \circ f$ is a chart of X compatible with the atlas of X. It is compatible when for any chart ϕ of X, the compositions $(\psi \circ f) \circ \phi^{-1} = \psi \circ f \circ \phi^{-1}$ and $\phi \circ (\psi \circ f)^{-1} = \phi \circ f^{-1} \circ \psi^{-1}$ are smooth. But this is exactly the condition that f and f^{-1} are smooth.

Conversely, if f is a diffeomorphism then it is bijective, and further the condition that $\psi \circ f$ is compatible with the atlas on X for every chart of Y is the same as the condition that f and f^{-1} are smooth.

In Class Exercises

16. Smooth maps on $\mathbb{S}^1$.

(a) Consider the function $f(x_1, x_2) = 2x_2 - x_1^2$. Using the stereographic projections ϕ_N and ϕ_S show that it is a smooth map $\mathbb{S}^1 \to \mathbb{R}$. Visualise this function.

(b) Visualise the function $g(x_1, x_2) = (x_2 + 1)^2 - 2$ on $\mathbb{S}^1$. Explain the connection to the previous function.

(c) Consider the antipodal map $A : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $x \mapsto -x$. Show it is smooth. Interpret this map geometrically.

(d) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Argue that $f = F|_{\mathbb{S}^1}$ is a smooth function. Hint. Consider $F = \Pi_i$ first.

(e) Are there any smooth functions on $\mathbb{S}^1$ that aren't of this form?

Solution.

(a) We write f in coordinates. This means using charts:

$$\mathrm{id} \circ f \circ \phi_N^{-1}(y) = f\left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right) = \frac{2y}{\|y\|^2 + 1} - \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}\right)^2,$$
$$\mathrm{id} \circ f \circ \phi_S^{-1}(y) = f\left(-\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right) = \frac{2y}{\|y\|^2 + 1} - \left(-\frac{\|y\|^2 - 1}{\|y\|^2 + 1}\right).$$

These are ugly, but smooth. Since every point belongs to the domain of either ϕ_N or ϕ_S, f is smooth at every point.

Here is a visualisation of this function `https://www.math3d.org/m3nQhJEgS`.

(b) This function is the same as the previous one, because $x_1^2 + x_2^2 = 1$ on $\mathbb{S}^1$ and

$$g = (x_2 + 1)^2 - 2 = x_2^2 + 2x_2 - 1 = 1 - x_1^2 + 2x_2 - 1 = f.$$

This shows that two functions may actually be the same function when restricted to a submanifold.

(c) In general if you want to show a map $\mathbb{S}^1 \to \mathbb{S}^1$ is smooth, you need to write it in four sets of coordinates: $\phi_N \circ f \circ \phi_N^{-1}, \phi_N \circ f \circ \phi_S^{-1}, \phi_S \circ f \circ \phi_N^{-1}, \phi_S \circ f \circ \phi_S^{-1}$. But observe that $a(N) = S$ and $a(S) = N$, so we only need to write it in two sets of coordinates to ensure that every possibility for x and $A(x)$ are covered. Namely

$$\begin{aligned}
\phi_S \circ A \circ \phi_N^{-1} &: \mathbb{R} \to \mathbb{R} \\
\phi_S \circ A \circ \phi_N^{-1}(y) &= \phi_S \circ A \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1} \right) \\
&= \phi_S \left(-\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, -\frac{2y}{\|y\|^2 + 1} \right) \\
&= \frac{-\frac{2y}{\|y\|^2+1}}{1 - \frac{\|y\|^2-1}{\|y\|^2+1}} = -y,
\end{aligned}$$

and

$$\begin{aligned}
\phi_N \circ A \circ \phi_S^{-1} &: \mathbb{R} \to \mathbb{R} \\
\phi_N \circ A \circ \phi_S^{-1}(y) &= \phi_N \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{-2y}{\|y\|^2 + 1} \right) \\
&= \frac{-\frac{2y}{\|y\|^2+1}}{1 - \frac{\|y\|^2-1}{\|y\|^2+1}} = -y.
\end{aligned}$$

Both of these are smooth.

This map sends each point on the circle to its opposite, its antipode.

(d) The calculation that Π_i is a smooth function on $\mathbb{S}^1$ is basically the same calculation as Exercise 11(e). Therefore we can write $f(x) = F(\Pi_1|_{\mathbb{S}^1}(x), \Pi_2|_{\mathbb{S}^1}(x))$ as the composition of smooth maps.

(e) No, but it is a bit tricky to prove. Take a smooth function of compact support $\chi : [0.5, 1.5] \to \mathbb{R}$ such that $\chi(1) = 1$. Define $F(x) = \chi(\|x\|)f(\hat{x})$ for $\|x\| \in [0.5, 1.5]$ and $F(x) = 0$ otherwise. I leave it to you to prove that this is a smooth function $\mathbb{R}^n \to \mathbb{R}$.

17. Smooth maps on $\mathbb{R}/\mathbb{Z}$.

(a) Show that $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}, [x] \mapsto \sin 2\pi x$ is a well defined function. Show further that it is a smooth function.

(b) Prove that functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ are equivalent to functions $F : \mathbb{R} \to \mathbb{R}$ with $F(x + 1) = F(x)$ for all $x \in \mathbb{R}$.

(c) Prove further that a function on $\mathbb{R}/\mathbb{Z}$ is smooth if and only if its periodic version is smooth.

(d) Generalise this result to maps $\mathbb{R}/\mathbb{Z} \to Y$.

Solution.

(a) If $x+n \in [x]$ then $\sin 2\pi(x+n) = \sin(2\pi x + 2\pi n) = \sin 2\pi x$. Therefore the definition of f is independent of the choice of representative. In other words, it is well defined.

To show it is smooth at $[x]$, use the chart ϕ_x. Then $\mathrm{id} \circ f \circ \phi_x^{-1} : (x-0.5, x+0.5) \to \mathbb{R}$ is just $t \mapsto \sin 2\pi t$. This is smooth.

(b) Suppose we have a periodic function $F : \mathbb{R} \to \mathbb{R}$. Define $f([x]) = F(x)$. This is well defined because if $x+n$ is another representative of $[x]$ then $F(x+n) = F(x+(n-1)) = \cdots = F(x)$ gives the same output. Conversely, given a function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ define $F = f \circ p$. Then $F(x+1) = f([x+1]) = f([x]) = F(x)$ is periodic.

(c) See next part.

(d) Nothing about part (b) relied on the target space being $\mathbb{R}$. So it immediately generalises. The relationship $F = f \circ p$ will be useful.

Choose any point $x \in \mathbb{R}$ or $[x] \in \mathbb{R}/\mathbb{Z}$. Let ψ be a chart of Y near $f([x]) = F(x)$. Then

$$\psi \circ f \circ \phi_x^{-1} = \psi \circ f \circ p = \psi \circ F = \psi \circ F \circ \mathrm{id}^{-1}$$

shows that f is smooth if and only if F is smooth.

18. A partition of unity for the interval $(0, 4)$.

We consider the open interval $M = (0, 4)$ as a 1-dimensional manifold. Take an open cover of M:

$$U_1 := (0,2)\ , \quad U_2 := (1,3)\ , \quad \text{and} \quad U_3 := (2,4).$$

(a) Give an example of three functions $f_1, f_2, f_3 \in C^\infty(M)$ with these properties:

$$0 \le f_k \le 1, \quad \mathrm{supp}(f_k) \subset U_k, \quad f_1 + f_2 + f_3 = 1.$$

(The support of a function is defined to be the closure of the points on which it is non-zero, $\mathrm{supp}(f_k) := \overline{\{\, x \in M \mid f_k(x) \neq 0 \,\}} \subset M$.) These functions form a partition of unity for M (Definition 1.26).

(b) Theorem 1.27 is even stronger! What additional property does the partition of unity given by Theorem 1.27 have, which our example does not have?

(c) Is it possible to have a partition of unity of M with this additional property and which has only finitely many functions (f_k)?

Solution.

(a) Suppose that we could find a smooth function g that was constant 1 on $(2-a, 2+a)$ for $0 < a < 0.5$, constant 0 outside $(1+a, 3-a)$ and valued in $[0,1]$. Then $f_1 = 1-g, f_2 = g, f_3 = 1-g$ would be an example of the sort we want.

There is a few standard ways to construct such smooth functions, often called bump, hat, cut-off, or plateau functions by various authors. In "Beweis der Existenz der Zerlegung der Eins" Martin gives one. Here is another. We begin with the basic example of a non-analytic function $A(x) = \exp(-x^{-1})$ for $x > 0$ and $A(x) = 0$ for $x \leq 0$. The function $A(x)A(1-x)$ is then smooth and zero outside $(0,1)$. For many purposes this function is already useful. Let $I(x) = \int_0^x A(t)A(1-t)\, \mathrm{d}t$ and $B(x) = I(x)/I(1)$. Then $B(x)$ is a smooth function that is constant 0 for $x \leq 0$ and constantly 1 for $x \geq 1$. In other words, it is a smooth function that 'joins' two constant functions.

We can take then

$$g(x) = B\left(\frac{x-(1+a)}{(2-a)-(1+a)}\right) + B\left(\frac{x-(3-a)}{(2+a)-(3-a)}\right).$$

`https://www.desmos.com/calculator/fdhywa4enf.`

(b) The stronger property that the partition in Theorem 1.27 has is that the supports of the f_k are compact, not just closed. For our example, $\mathrm{supp}(f_1) = \overline{(0, 2-a)} = (0, 2-a]$ is closed in M (note, the closure is taken in the manifold M). This is not compact, because the open cover $\{(n^{-1}, 2)\}_{n\in\mathbb{N}}$ has no finite subcover.

(c) No. By the definition of partition of unity $M = \bigcup \mathrm{supp}(f_k)$. If there were only finitely many f_k and their supports were compact, then M would be the finite union of compact sets, thus compact. But M is not compact.

Additional Exercises

19. Diffeomorphism as an equivalence.

In Exercise 13 you gave two incompatible atlases on the topological space $\mathbb{R}$. Therefore we have two manifolds: the standard one $(\mathbb{R}, \mathcal{A})$ and your example $(\mathbb{R}, \tilde{\mathcal{A}})$. Show these two manifolds are diffeomorphic.

Why is diffeomorphism an equivalence relation on manifolds?

This leads to the question: on a topological space X how many manifold structures exist that are mutually non-diffeomorphic? Often these are called 'exotic' manifold structures.

Solution. I will use my example $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 0, \\ 2x & \text{if } x > 0. \end{cases}$$

The two atlases are $\mathcal{A} = \{\text{id}\}$ and $\tilde{\mathcal{A}} = \{f\}$. I claim that $f : (\mathbb{R}, \tilde{\mathcal{A}}) \to (\mathbb{R}, \mathcal{A})$ is a diffeomorphism. We know it is a homeomorphism from Exercise 13. But this seems strange because it doesn't look like f is differentiable. However we need to interpret this in the sense of manifolds: f is smooth as a map between manifolds if

$$\text{id} \circ f \circ f^{-1} : \mathbb{R} \to \mathbb{R}$$

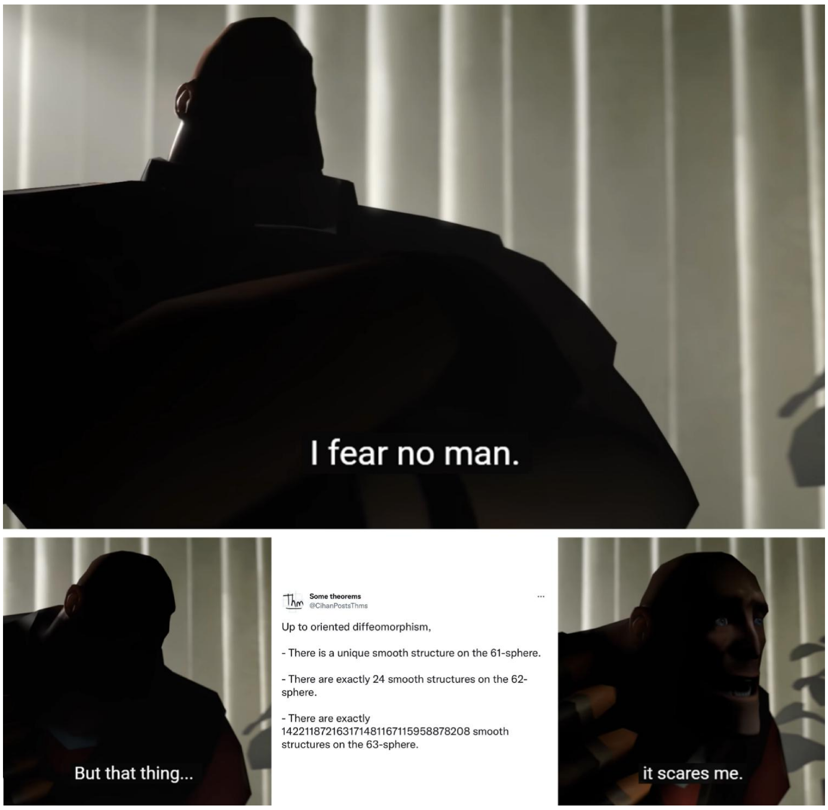
is smooth in the Euclidean sense. This is just the identity function, so it is smooth. The inverse is also the identity. Hence f is a diffeomorphism between these two manifolds.

Every manifold is diffeomorphic to itself using the identity map. The inverse of a diffeomorphism is a diffeomorphism, so the relationship is symmetric. And the composition of diffeomorphisms is again a diffeomorphism, which gives transitivity.

Up to diffeomorphism, there is only one manifold structure on $\mathbb{R}$. That's how I could be confident your example from Exercise 13 would be diffeomorphic to the standard atlas. The only Euclidean space $\mathbb{R}^n$ that has exotic manifold structures is $\mathbb{R}^4$. The list for spheres is

Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
Smooth types	1	1	1	$\geq$1	1	1	28	2	8	6	992	1	3	2	16256	2	16

It is completely known except for $n = 4$.



Terminology

glatt = smooth.

Zerlegung der Eins = partition of unity.

Träger = support (symbol is supp).