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Analysis III
3. Exercise: Smooth Maps

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Preparation Exercises

14. Properties of Smooth Maps.

(a) Let $x \in X$ be a point that is in the domain of two charts $\phi_1 : U_1 \to \phi[U_1]$ and $\phi_2 : U_2 \to \phi[U_2]$. Let $f : X \to Y$ be a map. Show that whether f is smooth at x does not depend on the chart.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Prove that $g \circ f$ is a smooth map.

(c) Show that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth in the sense of Definition 1.22 (as a map between manifolds), exactly if it is smooth in the Euclidean sense.

(d) Let X be a manifold and $\phi : U \to \phi[U] \subset \mathbb{R}^n$ a chart. Explain why $\phi : U \to \mathbb{R}^n$ is a smooth map in the sense of manifolds.

(e) Let $f : X \to \mathbb{R}$ be a smooth function (we often use the word function for maps to $\mathbb{R}$, though they are interchangable). Choose a chart $\phi : U \to \phi[U] \subset \mathbb{R}^n$. Define for $1 \leq i \leq n$ the i^{th}-partial derivative of f with respect to ϕ as

$$\frac{\partial f}{\partial \phi_i} : U \to \mathbb{R}, \quad x \mapsto \frac{\partial (f \circ \phi^{-1})}{\partial y_i}(\phi(x)).$$

That that this is a smooth function on U. It is sometimes written as $\partial f / \partial \phi_{,i}$ (especially in physics) to make clear that the index refers to the coordinate and not which chart from the atlas.

15. Diffeomorphism.

Let X, Y be differential manifolds. Show that X and Y are diffeomorphic (Def 1.21) exactly when there is a bijective smooth map $f : X \to Y$ whose inverse is also smooth.

In Class Exercises

16. Smooth maps on $\mathbb{S}^1$.

(a) Consider the function $f(x_1, x_2) = 2x_2 - x_1^2$. Using the stereographic projections ϕ_N and ϕ_S show that it is a smooth map $\mathbb{S}^1 \to \mathbb{R}$. Visualise this function.

(b) Visualise the function $g(x_1, x_2) = (x_2 + 1)^2 - 2$ on $\mathbb{S}^1$. Explain the connection to the previous function.

(c) Consider the antipodal map $A : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $x \mapsto -x$. Show it is smooth. Interpret this map geometrically.

(d) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Argue that $f = F|_{\mathbb{S}^1}$ is a smooth function. Hint. Consider $F = \Pi_i$ first.

(e) Are there any smooth functions on $\mathbb{S}^1$ that aren't of this form?

17. Smooth maps on $\mathbb{R}/\mathbb{Z}$.

(a) Show that $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}, [x] \mapsto \sin 2\pi x$ is a well defined function. Show further that it is a smooth function.

(b) Prove that functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ are equivalent to functions $F : \mathbb{R} \to \mathbb{R}$ with $F(x + 1) = F(x)$ for all $x \in \mathbb{R}$.

(c) Prove further that a function on $\mathbb{R}/\mathbb{Z}$ is smooth if and only if its periodic version is smooth.

(d) Generalise this result to maps $\mathbb{R}/\mathbb{Z} \to Y$.

18. A partition of unity for the interval $(0, 4)$.

We consider the open interval $M = (0, 4)$ as a 1-dimensional manifold. Take an open cover of M:

$$U_1 := (0, 2)\ , \quad U_2 := (1, 3)\ , \quad \text{and} \quad U_3 := (2, 4).$$

(a) Give an example of three functions $f_1, f_2, f_3 \in C^\infty(M)$ with these properties:

$$0 \leq f_k \leq 1, \quad \operatorname{supp}(f_k) \subset U_k, \quad f_1 + f_2 + f_3 = 1.$$

(The support of a function is defined to be the closure of the points on which it is non-zero, $\operatorname{supp}(f_k) := \overline{\{\, x \in M \mid f_k(x) \neq 0 \,\}} \subset M$.) These functions form a partition of unity for M (Definition 1.26).

(b) Theorem 1.27 is even stronger! What additional property does the partition of unity given by Theorem 1.27 have, which our example does not have?

(c) Is it possible to have a partition of unity of M with this additional property and which has only finitely many functions (f_k)?

Additional Exercises

19. Diffeomorphism as an equivalence.

In Exercise 13 you gave two incompatible atlases on the topological space $\mathbb{R}$. Therefore we have two manifolds: the standard one $(\mathbb{R}, \mathcal{A})$ and your example $(\mathbb{R}, \tilde{\mathcal{A}})$. Show these two manifolds are diffeomorphic.

Why is diffeomorphism an equivalence relation on manifolds?

This leads to the question: on a topological space X how many manifold structures exist that are mutually non-diffeomorphic? Often these are called 'exotic' manifold structures.

Terminology

glatt = smooth.
Zerlegung der Eins = partition of unity.
Träger = support (symbol is supp).