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Analysis III
2. Exercise: Charts

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Preparation Exercises

8. Properties of charts. Let X be a manifold with atlas $\mathcal{A}$. Let $\phi : U \to \phi[U] \subset \mathbb{R}^n$ be a chart of $\mathcal{A}$.

(a) State the definition of a chart and what it means for two charts to be compatible.

(b) Is compatibility of charts an equivalence relation?

(c) Restriction: If V is an open subset of U, prove $\psi := \phi|_V : V \to \mathbb{R}^n$ is a chart of X that is compatible with ϕ.

(d) Composition: If $F : \phi[U] \to W \subset \mathbb{R}^n$ is a diffeomorphism, prove $\psi := F \circ \phi : U \to W$ is a chart of X that is compatible with ϕ.

(e) Show that every point $x \in X$ has a neighbourhood V such that there is a chart $\psi : V \to B(0,1) \subset \mathbb{R}^n$ with $\psi(x) = 0$ that is compatible with $\mathcal{A}$.

Solution.

(a) A chart of a topological space X is a homeomorphism of an open subset of X to an open subset of $\mathbb{R}^n$.

If $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ are charts of X, then they are compatible if

$$\psi|_{U\cap V} \circ \phi^{-1}|_{\phi[U\cap V]} : \phi[U \cap V] \to \psi[U \cap V] \text{ and}$$
$$\phi|_{U\cap V} \circ \psi^{-1}|_{\psi[U\cap V]} = (\psi|_{U\cap V} \circ \phi^{-1}|_{\phi[U\cap V]})^{-1} : \psi[U \cap V] \to \phi[U \cap V]$$

are smooth maps. It makes sense to ask whether they are smooth because they are maps between subsets of $\mathbb{R}^n$. These maps are called change of coordinates maps. We often leave out the restrictions because they can be determined from context. Another way to state this is that the change of coordinates must be a diffeomorphism.

(b) It is not. It is reflexive and symmetric, but it is not in general transitive.

(c) Since V is a subset of U, the intersection $U \cap V = V$ is simple. The change of coordinates map is

$$\phi|_{U\cap V} \circ \psi^{-1}|_{\psi[U\cap V]} = \phi|_V \circ \psi^{-1}|_{\psi[V]} = \psi \circ \psi^{-1} = \mathrm{id}.$$

This is smooth, and the inverse is also id and smooth.

(d) In this case the two domains of the charts are equal, so $U \cap U = U$. The change of coordinates map is

$$\psi|_{U\cap V} \circ \phi^{-1}|_{\phi[U\cap V]} = \psi \circ \phi^{-1} = F \circ \phi \circ \phi^{-1} = F.$$

By assumption, F is a diffeomorphism.

(e) We combine the two previous parts. For any point x it belongs to a coordinate chart $\phi : U \to \phi[U]$. Because ϕ is a homeomorphism, $\phi[U]$ is open in $\mathbb{R}^n$. That means we can find a ball $B(\phi(x), \varepsilon) \subset U$. Let $V = \phi^{-1}[B(\phi(x), \varepsilon)]$. We know from part (c) that $\phi|_V : V \to B(\phi(x), \varepsilon)$ is a compatible chart. Obverse that $F(y) = \varepsilon^{-1}(y - \phi(x)) : B(\phi(x), \varepsilon) \to B(0, 1)$ is a diffeomorphism. Then $\psi = F \circ \phi|_V$ is the chart we want, which is compatible by part (d).

We say that a chart with the property $\psi(x) = 0$ is *centered on x*.

9. Graphs as manifolds.

Consider the parabola $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1^2\}$ as a topological space, with the subspace topology from $\mathbb{R}^2$. In this question we will give it a manifold structure. This means, we give it an atlas. This example falls within Beispiel 1.18(iv) of the script, but try to prove it by hand.

Let $\Pi_1(x_1, x_2) = x_1, \Pi_2(x_1, x_2) = x_2$ be the projections $\mathbb{R}^2 \to \mathbb{R}$ onto the coordinate axes.

(a) How do we know that X is Hausdorff and Lindelöf?

(b) Let $\phi_1 := \Pi_1|_X$. Is this a chart for X?

(c) Is $\mathcal{A} := \{\phi_1\}$ an atlas for X?

(d) Let $U_2 := X \cap \{x_1 > 0\}$ and $\phi_2 := \Pi_2|_{U_2}$. Prove ϕ_2 is a chart that is compatible with $\mathcal{A}$.

Solution.

(a) X is a closed subset of $\mathbb{R}^2$. It is Hausdorff and Lindelöf by Exercise 4(c).

(b) The domain of ϕ_1 is X, which is by definition an open subset of X. We must show that ϕ_1 is a homeomorphism. The inverse of $\phi_1(x_1, x_2) = x_1$ is $\phi_1^{-1}(y) = (y, y^2)$. This shows that it is bijective.

It is easy to see that the projections are continuous, since the inverse image $\Pi_1[U] = U \times \mathbb{R}$ of any open set $U \subset \mathbb{R}$ is open (or because they are polynomial, and use a result of Analysis II). The restriction of a continuous map is also continuous. The inverse map is also continuous.

(c) The domain of ϕ_1 is X, so the whole manifold is covered. Every chart is compatible with itself. Therefore yes, $\mathcal{A}$ is an atlas for X.

(d) U_2 is an open set of X by the definition of subspace topology. The reason for the restriction $\{x_1 > 0\}$ is to make ϕ_2 a bijection from U_2 to $\mathbb{R}_+$. Indeed, the inverse is $\phi_2^{-1}(y) = (+\sqrt{y}, y)$. ϕ_2 and ϕ_2^{-1} are continuous by Analysis II results.

To be compatible with an atlas means to be compatible with every chart in it. Hence we need to check that ϕ_1 and ϕ_2 are compatible with on another. The intersection of their domains is $X \cap U_2 = U_2$. The change of coordinates map are

$$\phi_2 \circ \phi_1^{-1} : \mathbb{R}_+ \to \mathbb{R}_+ \qquad y \mapsto \phi_2 \circ \phi_1^{-1}(y) = \phi_2(y, y^2) = y^2,$$
$$\phi_1 \circ \phi_2^{-1} : \mathbb{R}_+ \to \mathbb{R}_+ \qquad y \mapsto \phi_1 \circ \phi_2^{-1}(y) = \phi_1(+\sqrt{y}, y) = +\sqrt{y}.$$

These are both smooth functions (the square root is smooth because we restricted y to be strictly positive).

In Class Exercises

10. Open subsets of a manifold.

Let $Y \subset X$ be an open subset of the manifold X with atlas $\mathcal{A}$. Give Y a manifold structure.

Solution. By a previous exercise, the open sets of Y are the open sets of X that are contained in Y. Since the restriction of a homeomorphism is a homeomorphism onto its image, for any chart $\phi : U \to \phi[U]$ of X the restriction $\phi|_Y : Y \cap U \to \phi[Y \cap U]$ is also a chart. The collection of these restricted charts make an atlas, as we have seen that restriction does not affect compatibility.

11. Stereographic projection is compatible with regular projection.

Consider the sphere from Example 1.18(iii) in the lecture script. We take $n = 1$ to get the circle $\mathbb{S}^1 = \{x_0^2 + x_1^2 = 1\}$. We defined two charts $\phi_N : \mathbb{S}^1 \setminus \{e_0\} \to \mathbb{R}$ and $\phi_S : \mathbb{S}^1 \setminus \{-e_0\} \to \mathbb{R}$, showed they were compatible, and that they form an atlas $\mathcal{A}_{\text{stereo}}$.

On the other hand, we can use the usual projections to get charts for $\mathbb{S}^1$. Let

$$\phi_1 := \Pi_0|_{\mathbb{S}^1 \cap \{x_1 > 0\}},\ \phi_2 := \Pi_0|_{\mathbb{S}^1 \cap \{x_1 < 0\}},\ \phi_3 := \Pi_1|_{\mathbb{S}^1 \cap \{x_0 > 0\}},\ \phi_4 := \Pi_1|_{\mathbb{S}^1 \cap \{x_0 < 0\}},$$

(a) Write out the vector formulae from the script in the case $n = 1$ in coordinates to get

$$\phi_N(x_0, x_1) = \frac{x_1}{1 - x_0}, \qquad \phi_S(x_0, x_1) = \frac{x_1}{1 + x_0}$$
$$\phi_N^{-1}(y) = \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right), \qquad \phi_S^{-1}(y) = \left(-\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right).$$

(b) Show that ϕ_1 is a chart of $\mathbb{S}^1$.

(c) Show that ϕ_3 is compatible with ϕ_1.

(d) Is ϕ_2 compatible with ϕ_1?

(e) Show that ϕ_1 is compatible with ϕ_S.

Solution.

(a)

$$\begin{aligned}\phi_N(x_0, x_1) &= e_0 + \frac{x - e_0}{1 - \langle x, e_0\rangle} \\ &= (1,0) + \frac{(x_0 - 1, x_1)}{1 - x_0} \\ &= \frac{(x_0 - 1 + 1 - x_0, x_1)}{1 - x_0} \\ &= \frac{(0, x_1)}{1 - x_0} \\ &= \frac{x_1}{1 - x_0} \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\phi_N^{-1}(y) &= \frac{(\|y\|^2 - 1)e_0 + 2y}{\|y\|^2 + 1} \\ &= \frac{(\|y\|^2 - 1, 2y)}{\|y\|^2 + 1} \\ &= \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right)\end{aligned}$$

(b) This is very similar to Exercise 9. The inverse of $\phi_1 : \mathbb{S}^1 \cap \{x_2 > 0\} \to (-1,1)$ is given by $\phi_1^{-1} : (-1,1) \to \mathbb{S}^1 \cap \{x_2 > 0\}$,

$$\phi_1^{-1}(y) = (y, \sqrt{1 - y^2}).$$

This shows that it is bijective. The formulae are clearly continuous on the given domains.

(c) The intersection of the domains of the charts is $\mathbb{S}^1 \cap \{x_1 > 0, x_2 > 0\}$. In other words, the first quadrant of the circle. The change of coordinates map is

$$\phi_3 \circ \phi_1^{-1}(y) = \phi_3(y, \sqrt{1 - y^2}) = \sqrt{1 - y^2}.$$

This is a diffeomorphism on $y \in (0,1)$.

(d) This is somewhat a question about terminology. The definition requires us to restrict the charts to the intersection $(\mathbb{S}^1 \cap \{x_2 > 0\}) \cap (\mathbb{S}^1 \cap \{x_2 < 0\}) = \emptyset$. What does it mean to restrict a function to the empty set? In this situation, we say that the charts are compatible. Below is the reason that we say that the charts are compatible. It is not so important for this course, but perhaps you will find it interesting.

In the definition of a function, a function $f : X \to Y$ is a subset $F \subset X \times Y$ such that for all $x \in X$ there is exactly one $y \in Y$ with $(x, y) \in F$. When $X = \emptyset$, the product $\emptyset \times Y = \emptyset$ and so there is only one function $e_Y : \emptyset \to Y$ defined by $E_Y = \emptyset$. "$\forall x \in \emptyset\ \exists! y \in Y : (x, y) \in E_Y$" is vacuously true. e_Y is called the empty function to Y.

It therefore follows that both functions we must consider are the empty function $\emptyset \to \emptyset$. These functions are smooth because they are smooth at every point (again, vacuously true).

(e) The intersection of the two domains is $(\mathbb{S}^1 \cap \{x_2 > 0\}) \cap (\mathbb{S}^1 \setminus \{-e_0\}) = \mathbb{S}^1 \cap \{x_2 > 0\}$. We compute the composition $\phi_S \circ \phi_1^{-1} : (-1, 1) \to (-1, 1)$,

$$\begin{aligned} \phi_S \circ \phi_1^{-1}(y) &= \phi_S(y, \sqrt{1 - y^2}) \\ &= \frac{\sqrt{1 - y^2}}{1 + y} = \sqrt{1 - y} \end{aligned}$$

This is a diffeomorphism. Therefore the charts ϕ_S and ϕ_1 are compatible.

12. The manifold $\mathbb{R}/\mathbb{Z}$.

Recall from Exercise 6 the quotient space $\mathbb{R}/\mathbb{Z}$ and the quotient map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. In particular we saw for every open unit interval $I_x = (x - 0.5, x + 0.5)$ that $p|_{I_x}$ is a homeomorphism. For any $x \in \mathbb{R}$, let $U_x := p[I_x]$ and $\phi_x := (p|_{I_x})^{-1} : U_x \to I_x$.

(a) Is ϕ_x a chart of $\mathbb{R}/\mathbb{Z}$?

(b) Prove that ϕ_x and ϕ_{x+n} are compatible, where $n \in \mathbb{Z}$.

(c) Prove that ϕ_x and ϕ_y are compatible, for any $x, y \in \mathbb{R}$.

(d) Why is $\{\phi_x | x \in \mathbb{R}\}$ an atlas for $\mathbb{R}/\mathbb{Z}$.

Solution.

(a) Yes it is. In Exercise 6 we showed that p is an open map, so U_x is open. The inverse of a homeomorphism is a homeomorphism. Thus ϕ_x is a chart. You can see that we did all the hard work in Exercise 6!

(b) We need to compute the change of coordinates map. The domains of both maps are the same, namely $U_x = p[I_x] = p[I_{x+n}] = U_{x+n}$. We make the following observation: if $t \in (x - 0.5, x + 0.5) = I_x$ then $t + n \in (x + n - 0.5, x + n + 0.5) = I_{x+n}$. Therefore

$$\phi_{x+n} \circ \phi_x^{-1}(t) = \phi_{x+n}([t]) = \phi_{x+n}([t + n]) = t + n.$$

The function $t \mapsto t + n$ is a diffeomorphism, so these charts are compatible.

(c) Conceptually this question is the same as the previous question, but the calculation is a little more annoying to write down. The intersection of the domains is $p[I_x] \cap p[I_y] = \mathbb{R}/\mathbb{Z} \setminus \{[x + 0.5], [y + 0.5]\}$. The domain of the change of coordinates map is then

$$\phi_x^{-1}[\mathbb{R}/\mathbb{Z} \setminus \{[x + 0.5], [y + 0.5]\}] = I_x \setminus [y + 0.5].$$

Assume that $x - y \notin \mathbb{Z}$. The case where they do differ by an integer was handled in the previous part. Choose $m \in \mathbb{Z}$ so that $y+0.5+m \in I_x$. For $t \in (x-0.5, y+0.5+m)$ we see that $t - m \in I_y$. On the other hand, for $t \in (y+0.5+m, x+0.5)$ it is $t - m - 1$ that lies in I_y. We can therefore compute

$$\phi_y \circ \phi_x^{-1}(t) = \phi_y([t]) = \begin{cases} \phi_y([t - m]) = t - m, & \text{for } t \in (x - 0.5, y + 0.5 + m), \\ \phi_y([t - m - 1]) = t - m - 1, & \text{for } t \in (y + 0.5 + m, x + 0.5). \end{cases}$$

This map is smooth. The inverse map looks very similar and is also smooth.

(d) It is an atlas because the union of the domains of the charts cover $\mathbb{R}/\mathbb{Z}$ and all the charts are compatible (by parts (b) and (c)).

Additional Exercises

13. Non-compatible differentiable atlases.

Let $\mathcal{A}$ be the natural atlas of $\mathbb{R}$, namely $\mathcal{A} = \{\mathrm{id}_{\mathbb{R}}\}$. Find another atlas $\widetilde{\mathcal{A}}$ of $\mathbb{R}$ that is not compatible with $\mathcal{A}$. (Compare to Exercise 1.20 in the script.)

[Hint. It is possible to find such an atlas $\widetilde{\mathcal{A}} = \{f\}$ that contains only one chart.]

Solution. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq 0, \\ 2x & \text{if } x > 0. \end{cases}$$

This function is a homeomorphism, but it is not smooth (it is not differentiable at $x = 0$). Therefore the change of coordinates map $f \circ \mathrm{id}^{-1} = f$ is not smooth. This shows the two charts are not compatible.

Terminology

Definitionsbreich = domain.

dicht = dense.

Karte = chart.

Umkehrabbildung = inverse map.

verträglich = compatible.