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Analysis III
2. Exercise: Charts

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Preparation Exercises

8. Properties of charts. Let X be a manifold with atlas $\mathcal{A}$. Let $\phi : U \to \phi[U] \subset \mathbb{R}^n$ be a chart of $\mathcal{A}$.

(a) State the definition of a chart and what it means for two charts to be compatible.

(b) Is compatibility of charts an equivalence relation?

(c) Restriction: If V is an open subset of U, prove $\psi := \phi|_V : V \to \mathbb{R}^n$ is a chart of X that is compatible with ϕ.

(d) Composition: If $F : \phi[U] \to W \subset \mathbb{R}^n$ is a diffeomorphism, prove $\psi := F \circ \phi : U \to W$ is a chart of X that is compatible with ϕ.

(e) Show that every point $x \in X$ has a neighbourhood V such that there is a chart $\psi : V \to B(0, 1) \subset \mathbb{R}^n$ with $\psi(x) = 0$ that is compatible with $\mathcal{A}$.

9. Graphs as manifolds.

Consider the parabola $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1^2\}$ as a topological space, with the subspace topology from $\mathbb{R}^2$. In this question we will give it a manifold structure. This means, we give it an atlas. This example falls within Beispiel 1.18(iv) of the script, but try to prove it by hand.

Let $\Pi_1(x_1, x_2) = x_1, \Pi_2(x_1, x_2) = x_2$ be the projections $\mathbb{R}^2 \to \mathbb{R}$ onto the coordinate axes.

(a) How do we know that X is Hausdorff and Lindelöf?

(b) Let $\phi_1 := \Pi_1|_X$. Is this a chart for X?

(c) Is $\mathcal{A} := \{\phi_1\}$ an atlas for X?

(d) Let $U_2 := X \cap \{x_1 > 0\}$ and $\phi_2 := \Pi_2|_{U_2}$. Prove ϕ_2 is a chart that is compatible with $\mathcal{A}$.

In Class Exercises

10. Open subsets of a manifold.

Let $Y \subset X$ be an open subset of the manifold X with atlas $\mathcal{A}$. Give Y a manifold structure.

11. Stereographic projection is compatible with regular projection.

Consider the sphere from Example 1.18(iii) in the lecture script. We take $n = 1$ to get the circle $\mathbb{S}^1 = \{x_0^2 + x_1^2 = 1\}$. We defined two charts $\phi_N : \mathbb{S}^1 \setminus \{e_0\} \to \mathbb{R}$ and $\phi_S : \mathbb{S}^1 \setminus \{-e_0\} \to \mathbb{R}$, showed they were compatible, and that they form an atlas $\mathcal{A}_{\text{stereo}}$.

On the other hand, we can use the usual projections to get charts for $\mathbb{S}^1$. Let

$$\phi_1 := \Pi_0|_{\mathbb{S}^1 \cap \{x_1 > 0\}},\ \phi_2 := \Pi_0|_{\mathbb{S}^1 \cap \{x_1 < 0\}},\ \phi_3 := \Pi_1|_{\mathbb{S}^1 \cap \{x_0 > 0\}},\ \phi_4 := \Pi_1|_{\mathbb{S}^1 \cap \{x_0 < 0\}},$$

(a) Write out the vector formulae from the script in the case $n = 1$ in coordinates to get

$$\phi_N(x_0, x_1) = \frac{x_1}{1 - x_0}, \qquad \phi_S(x_0, x_1) = \frac{x_1}{1 + x_0}$$

$$\phi_N^{-1}(y) = \left(\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right), \qquad \phi_S^{-1}(y) = \left(-\frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \frac{2y}{\|y\|^2 + 1}\right).$$

(b) Show that ϕ_1 is a chart of $\mathbb{S}^1$.

(c) Show that ϕ_3 is compatible with ϕ_1.

(d) Is ϕ_2 compatible with ϕ_1?

(e) Show that ϕ_1 is compatible with ϕ_S.

12. The manifold $\mathbb{R}/\mathbb{Z}$.

Recall from Exercise 6 the quotient space $\mathbb{R}/\mathbb{Z}$ and the quotient map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. In particular we saw for every open unit interval $I_x = (x - 0.5, x + 0.5)$ that $p|_{I_x}$ is a homeomorphism. For any $x \in \mathbb{R}$, let $U_x := p[I_x]$ and $\phi_x := (p|_{I_x})^{-1} : U_x \to I_x$.

(a) Is ϕ_x a chart of $\mathbb{R}/\mathbb{Z}$?

(b) Prove that ϕ_x and ϕ_{x+n} are compatible, where $n \in \mathbb{Z}$.

(c) Prove that ϕ_x and ϕ_y are compatible, for any $x, y \in \mathbb{R}$.

(d) Why is $\{\phi_x | x \in \mathbb{R}\}$ an atlas for $\mathbb{R}/\mathbb{Z}$.

Additional Exercises

13. Non-compatible differentiable atlases.

Let $\mathcal{A}$ be the natural atlas of $\mathbb{R}$, namely $\mathcal{A} = \{\mathrm{id}_{\mathbb{R}}\}$. Find another atlas $\widetilde{\mathcal{A}}$ of $\mathbb{R}$ that is not compatible with $\mathcal{A}$. (Compare to Exercise 1.20 in the script.)

[Hint. It is possible to find such an atlas $\widetilde{\mathcal{A}} = \{f\}$ that contains only one chart.]

Terminology

Definitionsbreich = domain.

dicht = dense.

Karte = chart.

Umkehrabbildung = inverse map.

verträglich = compatible.