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Analysis III
1. Exercise: Topology

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You do not require a certain number of exercise points to be admitted to the exam. Regardless, exercises are one of the best way to learn mathematics and improve your understanding, and I encourage you to do as many of them as possible.

The exercises are divided into three types. “Preparation Exercises” should be attempted yourself (or with a partner) before the tutorial. Please submit these exercises as a pdf to

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the day before the tutorial. I will read and provide feedback. Even if you do not have a full solution, it’s very useful for me to see what you find difficult so I can make sure to cover it in the tutorial. There are also “In Class Exercises”, which we will try to solve together in the tutorial. Finally there are “Additional Exercises”. These might be exercises that go beyond the course or that give a deeper explanation.

Preparation Exercises

1. Continuity in metric spaces.

Exercise 1.7 in the script.
In this question we show that the ε-δ-definition of continuity in metric spaces agrees with the definition of continuity in topological spaces.

Let (X, d) and (X', d') be two metric spaces, and $f : X \to X'$ a map between them. Demonstrate the following are equivalent:

(1) For every open subset O' of X', the pre-image $f^{-1}[O']$ is open in X.

(2) For every point $p \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ so that for every point $q \in X$ with $d(p, q) < \delta$ it holds that $d'(f(p), f(q)) < \varepsilon$.

Solution. It is useful to restate (2) in terms of balls. $d(x, y) < R$ is equivalent to $y \in B(x, R)$. This is the definition of a ball in a metric space. So (2) is equivalent to
(2’) For every point $p \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ so that $f[B(p, \delta)] \subseteq B(f(p), \varepsilon)$.

Suppose that (1) is true. Choose any point $p \in X$ and any $\varepsilon > 0$. Consider the ball $O' = B(f(p), \varepsilon)$ in X'. By (1), we know that $O = f^{-1}[O']$ is an open set of X that contains p. Therefore there is a ball $B(p, \delta) \subseteq O$ for some $\delta > 0$. $f[B(p, \delta)] \subseteq f[O] = O' = B(f(p), \varepsilon)$.

Suppose that (2’) is true. Choose any open set O' of X' and let $O = f^{-1}[O']$. Choose any point $p \in O$. We need to show that there is a ball $B(p, \delta) \subseteq O$. But O' is open and contains $f(p)$, so we know that there exists a ball $B(f(p), \varepsilon) \subseteq O'$. (2') now guarantees the existence of such a $B(p, \delta)$, because $f[B(p, \delta)] \subseteq B(f(p), \varepsilon) \subseteq O'$ implies $B(p, \delta) \subseteq f^{-1}[O']$.

2. Homeomorphism.

Let $f : X \to Y$. Define what it means for f to be a homeomorphism (look this up if necessary). Show that this is equivalent to:

- f is bijective
- f is continuous
- f is an *open map*: For every open set $U \subset X$, the image $f[U]$ is open in Y.

Hint. Show for all subsets $A \subset X$ that $(f^{-1})^{-1}[A] = f[A]$.

Solution. By definition, a homeomorphism is a bijective map that is continuous and whose inverse is continuous. (This is given as an aside in Definition 1.15).

Two out of the three conditions are the same, so it remains to show that the third condition is equivalent (assuming the first two). Note that there is indeed an inverse $f^{-1} : Y \to X$, because f is bijective. Also note for any subset $A \subset X$ that

$$(f^{-1})^{-1}[A] = \{y \in Y \mid f^{-1}(y) \in A\} = \{y \in Y \mid y \in f[A]\} = f[A].$$

Suppose that $f^{-1} : Y \to X$ is continuous. This means for every open set $U \subset X$ that $(f^{-1})^{-1}[U] = f[U]$ is open. But this is the condition that f is an open map. Conversely, suppose that f is an open map. Then for every U we know $f[U] = (f^{-1})^{-1}[U]$ is open. But this is the condition that f^{-1} is continuous.

3. A Characterisation of connected spaces.

Let X be a metric space. Show that the following properties are equivalent:

(1) X is connected (Definition 1.8).

(2) There does not exists two non-empty open subsets U, V of X with $U \cup V = X$ and $U \cap V = \varnothing$.

Solution. Suppose that there are two open sets with $U \cup V = X$ and $U \cap V = \varnothing$. It follows that $U = X \setminus V$. Because V is open, U is closed. Hence U and V are both open and closed. If X is connected as per Definition 1.8, then one must be the empty set, which shows (2). On the other hand, if (1) is not true then take V to be an open and closed set that is not $\varnothing$ and not X. This shows (2) is not true either.

In Class Exercises

4. Closed and Open subsets of $\mathbb{R}^n$ are Hausdorff and Lindelöf.

In the lectures, a manifold was defined as a Hausdorff and Lindelöf topological space together with an atlas. Here are facts that make it easy to check the topological properties:
(1) Beispiel 1.2(iv): Every subset of a metric space is a metric space.
(2) Every metric space is Hausdorff.
(3) Definition 1.28: A topological space is called *locally compact* when every point has a neighbourhood U so that $\overline{U}$ is compact.
(4) Every open subset and every closed subset of a locally compact space is a locally compact space.
(5) Theorem 1.29(ii,iii): A locally compact Hausdorff space is Lindelöf if and only if it can be written as the countable union of compact sets.

(a) Explain why (1) is true.

(b) Why does it follow from these five facts that every closed subset of $\mathbb{R}^n$ is Hausdorff and Lindelöf?
[Hint. Let $K_n = \overline{B(0,n)}$ and notice $\mathbb{R}^n = \cup_{n\in\mathbb{N}} K_n$.]

(c) Prove that every open subset of $\mathbb{R}^n$ is Hausdorff and Lindelöf.

Solution.

(a) As it says in Beispiel 1.2(iv), let $A \subset X$ be a subset of a metric space (X, d). Let $\tilde{d}$ be the restriction of d to $A \times A$. To be a metric, you need that three properties hold for all points. Eg for all $x, y \in A$ we need $\tilde{d}(x,y) = \tilde{d}(y,x)$. But since it is just restriction, there's not really anything to check here

$$\tilde{d}(x,y) = d(x,y) = d(y,x) = \tilde{d}(y,x).$$

And the same for the other properties.

(b) Let the closed subset of $\mathbb{R}^n$ be called A. Because $\mathbb{R}^n$ is a metric space, by points (1) and (2) we know that A is Hausdorff. Next observe that $\mathbb{R}^n$ is locally compact: for any point $x \in \mathbb{R}^n$ take the closed ball $\overline{B(x,1)}$. From point (4), we know then that A is also locally compact.

Now that we know A is Hausdorff and locally compact, it remains to show, due to point (5), that it can be written as the union of countably many compact sets. Following the hint, define $A_n := K_n \cap A$. These sets are compact because they are the intersection of a compact set and a closed set. Finally

$$\bigcup A_n = \bigcup (A \cap K_n) = A \cap \bigcup K_n = A \cap \mathbb{R}^n = A$$

demonstrates A can written as the union of countably many compact sets.

(c) Similar to closed subsets, we see that an open subset $U \subset \mathbb{R}^n$ is locally compact and Hausdorff. It remains to write U as the countable union of compact sets. For each point $x \in U$, there is an open ball $B(x, r_x) \subset U$. By reducing the radius slightly, we can assume that r_x is rational. Because we know the rational numbers are dense in $\mathbb{R}$, we can choose a rational point $q_x \in \mathbb{Q}^n$ such that $d(x, q_x) < 0.5r_x$. Then $x \in \overline{B(q_x, 0.5r_x)} \subset U$. This collection of closed balls $\{\overline{B(q_x, 0.5r_x)} | x \in U\}$ is a subset of the countable set $\{\overline{B(q, 0.5r)} | q \in \mathbb{Q}^n, r \in \mathbb{Q}\}$ and is therefore countable. Thus we have written U as a countable union of compact sets.

5. The subspace topology.

Let $A \subset X$ be a subset of a topological space (X, τ). Define the *subspace topology* $\tau_A = \{A \cap U | U \in \tau\}$.

(a) Prove this is a topology on A.

(b) Suppose A is an open subset of X. Show that $V \in \tau_A$ if and only if $V \in \tau$ and $V \subset A$.

(c) Show that B closed set of (A, τ_A) if and only if it the intersection of a closed set of X with A.

(d) Let $X = \mathbb{R}$ and $A = [0, 1)$. Give an example of an open set in τ_A that is not in τ.

(e) Let $A = \{0\} \cup \{n^{-1} \mid n \in \mathbb{N}_+\} \subset \mathbb{R}$. What are the connected components of A (in the subspace topology)?

Solution.

(a) There are three properties we need to check. First $A = A \cap X$ and $\emptyset = A \cap \emptyset$ shows that $A, \emptyset \in \tau_A$. Second, suppose we have a collection of open sets $V_\alpha = A \cap U_\alpha$. Then

$$\bigcup_\alpha V_\alpha = \bigcup_\alpha (A \cap U_\alpha) = A \cap \left(\bigcup_\alpha U_\alpha \right)$$

shows that the union also belongs to τ_A. The same reasoning shows that the finite intersection of open set is an open set of A.

(b) Suppose that $V \in \tau_\alpha$, so $V = A \cap U$. But if A is open in X, then the intersection V is also open in X. Conversely, if $V \in \tau$ and $V \subset A$ then by definition $V = A \cap V \in \tau_A$.

(c) By definition, a closed set is the complement of an open set. Suppose that B is $A \setminus V$ and $V = A \cap U$. Then

$$A \cap (X \setminus U) = (A \cap X) \setminus (A \cap U) = A \cap V.$$

This calculation also shows that $A \cap (X \setminus U)$ is always a closed subset of A.

(d) $[0, 0.5)$ is not an open set in $\mathbb{R}$. But it is an open set in $[0, 1)$ because $[0, 0.5) = [0, 1) \cap (-1, 0.5)$.

(e) The connected components of A are just the singleton sets (subsets that contain a single point).

To explain this answer, let us understand the topology of A. Consider the point $n^{-1} \in A$. Then $\{n^{-1}\} = A \cap B(n^{-1}, n^{-1}(n+1)^{-1})$ shows that it is an open set. However it is also a closed set, because $\{n^{-1}\} = A \cap \overline{B(n^{-1}, 0.5n^{-1}(n+1)^{-1})}$.

Suppose that Y is a connected set of A that has two or more points in it. Then it must contain n^{-1} for some n. We see that $\{n^{-1}\}$ is both closed and open in Y. But then we can write $Y = (Y \setminus \{n^{-1}\}) \cup \{n^{-1}\}$ as the union of two open and disjoint sets. This is a contradiction due to Exercise 2. We conclude that the only connected sets of A are the singleton sets, so these must be the connected components.

6. The quotient topology.

Let X be a topological space and $\sim$ an equivalence relation. Recall that $[x] := \{y \in X \mid y \sim x\}$ is called the equivalence class of x. Every point of X belongs to exactly one equivalence class. Define $X/\sim$ to be the set of equivalence classes. This is called the quotient space. There is a surjective function $p : x \mapsto [x]$ called the quotient map or quotient projection. We define a topology on $X/\sim$ by

$$\tau_\sim = \{V \in \mathcal{P}(X) \mid p^{-1}[V] \text{ is open in } X\}.$$

(a) Prove this is a topology.

(b) Prove that p is continuous.

(c) Show that $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$ is an equivalence relation on $\mathbb{R}$. We usually call the quotient space $\mathbb{R}/\mathbb{Z}$.

(d) Show that if U is open in $\mathbb{R}$ then $p[U]$ is open in $\mathbb{R}/\mathbb{Z}$.

(e) Choose any point $[x] \in \mathbb{R}/\mathbb{Z}$. Let $I = (x - 0.5, x + 0.5) \subset \mathbb{R}$ be an interval and $U = p[I]$. Prove that $p|I : I \to U$ is a homeomorphism.

Solution.

(a) Just like the proof the subspace topology follows from the set laws about intersections, this follows from the set laws about preimages. If V_α are open sets, then

$$p^{-1}\left[\bigcup_\alpha V_\alpha\right] = \bigcup_\alpha p^{-1}[V_\alpha]$$

shows that the union is open. If there are finitely many then

$$p^{-1}\left[\bigcap_\alpha V_\alpha\right] = \bigcap_\alpha p^{-1}\left[V_\alpha\right]$$

shows that the intersection is open. Finally $p^{-1}[X/\sim] = X$ because p is surjective. X is open so $X/\sim \in \tau_\sim$. Likewise for $p^{-1}[\emptyset] = \emptyset$.

(b) By definition p is continuous if for every open set V in the quotient the set $p^{-1}[V]$ is open in X. But this is exactly the condition that defines whether V is open.

(c) Reflexivity: $x - x = 0 \in \mathbb{Z}$.
Symmetry: If $x - y \in \mathbb{Z}$ then $y - x = -(x - y) \in \mathbb{Z}$.
Transitivity: If $x - y, y - z \in \mathbb{Z}$ then $x - z = (x - y) + (y - z) \in \mathbb{Z}$.

(d) Let us first show how to calculate $p^{-1}[p[A]]$ for any subset $A \subset \mathbb{R}$. Suppose y belongs to this set. That means $p(y) = [y] \in p[A]$. There exists $a \in A$ with $p(y) = p(a)$. In other words they belong to the same equivalence class and $y = a + n$ for some $n \in \mathbb{Z}$. Conversely $p(a + n) = p(a)$ shows that $a + n \in p^{-1}[p[A]]$. In summary

$$p^{-1}[p[A]] = \{a + n \mid a \in A, n \in \mathbb{Z}\}.$$

This can help us see that $p[U]$ is open.

$$p^{-1}[p[U]] = \bigcup_{n\in\mathbb{Z}}(U + n).$$

This is the union of open sets of $\mathbb{R}$, since the translation of an open set U is still open. Therefore $p[U]$ is open in $\mathbb{R}/\mathbb{Z}$ for any open set $U \subset \mathbb{R}$.

(e) Homeomorphism means that it is bijective, continuous, and the inverse is continuous. We will however use the alternative criterion that it is bijective, continuous, and an open map.

First we demonstrate that $\tilde{p} : I \to U, y \mapsto p(y)$ is bijective. By definition it is surjective. If $p(y) = p(y')$ then $y - y' \in \mathbb{Z}$. But the difference of two points of $(x - 0.5, x + 0.5)$ is always strictly between -1 and 1. Therefore $y - y' = 0$. This shows $\tilde{p}$ is injective too.

We know it is continuous by part (b).

We know it is an open map by part (d).

Additional Exercises

7. (Not) Hausdorff and Lindelöf Manifolds, the type of spaces we study in this course, are defined to be both Hausdorff and Lindelöf. In this question we give two examples: The 'line with two origins' is not Hausdorff and the 'long ray' is not Lindelöf. This is extra material to help you understand these properties.

(a) Let $D = \mathbb{R} \cup \{0'\}$. A set U is open in D if U is a subset of $\mathbb{R}$ and is open in $\mathbb{R}$, or if U contains the new point $0'$ and $U \cup \{0\} \setminus \{0'\}$ is open in $\mathbb{R}$. Show that the sequence $(n^{-1})_{n\in\mathbb{N}^+}$ has both 0 and $0'$ as limit points (the definition of convergence in a topological space is after Definition 1.6). The space D is called the 'line with two origins'.

(b) Consider the topological space $R := \mathbb{N} \times [0, 1)$ with the ordering $(m, x) < (n, y)$ if $m < n$, or $m = n$ and $x < y$. Give a function $f : R \to [0, \infty)$ that preserves the order relation.

(c) There exists a set Ω, called the first uncountable ordinal, with the following properties:

(1) it is uncountable

(2) it is *well-ordered*. A set is well-ordered when there is an order relation $<$ in which every non-empty subset has a minimum, a smallest element. $\mathbb{R}$ with the normal order is not well-ordered, for example $(0, 1)$ does not contain a minimum. $\mathbb{N}$ with the usual order is well-ordered.

(3) for every $a \in \Omega$, the subset $H(a) := \{b \in \Omega \mid b < a\}$ is countable.

Let $R' := \Omega \times [0, 1)$ with the ordering $(a, x) < (b, y)$ if $a < b$, or $a = b$ and $x \leq y$. Let 0_Ω be the minimum of Ω so that $O = (0_\Omega, 0)$ is the minimum of R'. An open interval in R' has the form $I(\alpha, \beta) := \{\phi \in R' \mid \alpha < \phi < \beta\}$ for $\alpha, \beta \in R'$ or $J(\beta) = \{O\} \cup I(O, \beta) = \{\phi \in R' \mid \phi < \beta\}$. Find an uncountable collection of open intervals such that no intervals intersect. Why is R' not Lindelöf?

R' is called the 'long ray' (R is called a ray, or half-line).

Solution.

(a) x is a limit point of a sequence (x_n) if and only if every open neighbourhood of x contains all but finitely many elements of (x_n).

Take any neighbourhood U of 0. It contains an interval of the form $(0, \varepsilon)$ for some $\varepsilon > 0$. By Archimedes' principle, there is a natural number $N > \varepsilon^{-1}$ and only the finitely many elements (n^{-1}) with $n < N$ do not lie in $(0, \varepsilon) \subset U$. Therefore 0 is a limit point of the sequence.

Take any neighbourhood U' of $0'$. By definition $U = U' \cup \{0\} \setminus \{0'\}$ is an open set of $\mathbb{R}$ containing 0. By the previous paragraph, it contains all but finitely many elements of the sequence. Therefore U' does too. This shows that 0 is also a limit point of the sequence.

(b) $f((m,x)) = m+x$. This preserves the order relation because if $m < n$ then $m+x < n+y$ for all $x, y \in [0,1)$ and if $m = n$ and $x < y$ then clearly $m+x < n+y$. It is also a bijection, with inverse given by $t \mapsto (\lfloor t \rfloor, t - \lfloor t \rfloor)$.

(c) It is easy to find such a collection: choose any uncountable subset $A \subset \Omega$ and consider the intervals $I((a,0),(a,0.5)) = \{(a,x) \mid 0 < x < 0.5\}$.

Consider the collection $\{J((a,0))\}_{a\in\Omega}$. This is a cover of R', but there is no countable subcover: Take any countable subset $I \subset \Omega$. By property (3), for every $a \in I$, $H(a)$ is countable. Then $H = \bigcup_{a\in I} H(a)$ is a countable union of countable sets, and so itself countable. Therefore $H \neq \Omega$, because Ω is uncountable. But if $(b,x) \in \bigcup_{a\in I}\{J((a,0))\}$ then $b \in H$. Because there are elements of Ω not in H, this shows the collection $\{J((a,0))\}_{a\in I}$ is not a cover.

If you are interested in these strange topological spaces, the famous reference is Steen and Seebach's Counterexamples in Topology. An online reference is the database website π-Base `https://topology.jdabbs.com/`.

Terminology

Bild = image.
Urbild = pre-image.
Homöomorphismus = homeomorphism.
Umgebung = neighbourhood.
unabzählbar = uncountable.
zusammenhängend = connected.