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Analysis III
1. Exercise: Topology

12 September 2022

You do not require a certain number of exercise points to be admitted to the exam. Regardless, exercises are one of the best way to learn mathematics and improve your understanding, and I encourage you to do as many of them as possible.

The exercises are divided into three types. "Preparation Exercises" should be attempted yourself (or with a partner) before the tutorial. Please submit these exercises as a pdf to

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the day before the tutorial. I will read and provide feedback. Even if you do not have a full solution, it's very useful for me to see what you find difficult so I can make sure to cover it in the tutorial. There are also "In Class Exercises", which we will try to solve together in the tutorial. Finally there are "Additional Exercises". These might be exercises that go beyond the course or that give a deeper explanation.

Preparation Exercises

1. Continuity in metric spaces.

Exercise 1.7 in the script.
In this question we show that the ε-δ-definition of continuity in metric spaces agrees with the definition of continuity in topological spaces.

Let (X, d) and (X', d') be two metric spaces, and $f : X \to X'$ a map between them. Demonstrate the following are equivalent:

(1) For every open subset O' of X', the pre-image $f^{-1}[O']$ is open in X.

(2) For every point $p \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ so that for every point $q \in X$ with $d(p, q) < \delta$ it holds that $d'(f(p), f(q)) < \varepsilon$.

2. Homeomorphism.

Let $f : X \to Y$. Define what it means for f to be a homeomorphism (look this up if necessary). Show that this is equivalent to:

- f is bijective
- f is continuous
- f is an *open map*: For every open set $U \subset X$, the image $f[U]$ is open in Y.

Hint. Show for all subsets $A \subset X$ that $(f^{-1})^{-1}[A] = f[A]$.

3. A Characterisation of connected spaces.

Let X be a metric space. Show that the following properties are equivalent:

(1) X is connected (Definition 1.8).

(2) There does not exists two non-empty open subsets U, V of X with $U \cup V = X$ and $U \cap V = \varnothing$.

In Class Exercises

4. Closed and Open subsets of $\mathbb{R}^n$ are Hausdorff and Lindelöf.

In the lectures, a manifold was defined as a Hausdorff and Lindelöf topological space together with an atlas. Here are facts that make it easy to check the topological properties:
(1) Beispiel 1.2(iv): Every subset of a metric space is a metric space.
(2) Every metric space is Hausdorff.
(3) Definition 1.28: A topological space is called *locally compact* when every point has a neighbourhood U so that $\overline{U}$ is compact.
(4) Every open subset and every closed subset of a locally compact space is a locally compact space.
(5) Theorem 1.29(ii,iii): A locally compact Hausdorff space is Lindelöf if and only if it can be written as the countable union of compact sets.

(a) Explain why (1) is true.

(b) Why does it follow from these five facts that every closed subset of $\mathbb{R}^n$ is Hausdorff and Lindelöf?
[Hint. Let $K_n = \overline{B(0, n)}$ and notice $\mathbb{R}^n = \cup_{n \in \mathbb{N}} K_n$.]

(c) Prove that every open subset of $\mathbb{R}^n$ is Hausdorff and Lindelöf.

5. The subspace topology.

Let $A \subset X$ be a subset of a topological space (X, τ). Define the *subspace topology* $\tau_A = \{A \cap U | U \in \tau\}$.

(a) Prove this is a topology on A.

(b) Suppose A is an open subset of X. Show that $V \in \tau_A$ if and only if $V \in \tau$ and $V \subset A$.

(c) Show that B closed set of (A, τ_A) if and only if it the intersection of a closed set of X with A.

(d) Let $X = \mathbb{R}$ and $A = [0, 1)$. Give an example of an open set in τ_A that is not in τ.

(e) Let $A = \{0\} \cup \{n^{-1} \mid n \in \mathbb{N}_+\} \subset \mathbb{R}$. What are the connected components of A (in the subspace topology)?

6. The quotient topology.

Let X be a topological space and $\sim$ an equivalence relation. Recall that $[x] := \{y \in X \mid y \sim x\}$ is called the equivalence class of x. Every point of X belongs to exactly one equivalence class. Define $X/\sim$ to be the set of equivalence classes. This is called the quotient space. There is a surjective function $p : x \mapsto [x]$ called the quotient map or quotient projection. We define a topology on $X/\sim$ by

$$\tau_\sim = \{V \in \mathcal{P}(X) \mid p^{-1}[V] \text{ is open in } X\}.$$

(a) Prove this is a topology.

(b) Prove that p is continuous.

(c) Show that $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$ is an equivalence relation on $\mathbb{R}$. We usually call the quotient space $\mathbb{R}/\mathbb{Z}$.

(d) Show that if U is open in $\mathbb{R}$ then $p[U]$ is open in $\mathbb{R}/\mathbb{Z}$.

(e) Choose any point $[x] \in \mathbb{R}/\mathbb{Z}$. Let $I = (x - 0.5, x + 0.5) \subset \mathbb{R}$ be an interval and $U = p[I]$. Prove that $p|I : I \to U$ is a homeomorphism.

Additional Exercises

7. (Not) Hausdorff and Lindelöf Manifolds, the type of spaces we study in this course, are defined to be both Hausdorff and Lindelöf. In this question we give two examples: The 'line with two origins' is not Hausdorff and the 'long ray' is not Lindelöf. This is extra material to help you understand these properties.

(a) Let $D = \mathbb{R} \cup \{0'\}$. A set U is open in D if U is a subset of $\mathbb{R}$ and is open in $\mathbb{R}$, or if U contains the new point $0'$ and $U \cup \{0\} \setminus \{0'\}$ is open in $\mathbb{R}$. Show that the sequence $(n^{-1})_{n \in \mathbb{N}^+}$ has both 0 and $0'$ as limit points (the definition of convergence in a topological space is after Definition 1.6). The space D is called the 'line with two origins'.

(b) Consider the topological space $R := \mathbb{N} \times [0, 1)$ with the ordering $(m, x) < (n, y)$ if $m < n$, or $m = n$ and $x < y$. Give a function $f : R \to [0, \infty)$ that preserves the order relation.

(c) There exists a set Ω, called the first uncountable ordinal, with the following properties:

(1) it is uncountable

(2) it is *well-ordered*. A set is well-ordered when there is an order relation $<$ in which every non-empty subset has a minimum, a smallest element. $\mathbb{R}$ with the normal order is not well-ordered, for example $(0,1)$ does not contain a minimum. $\mathbb{N}$ with the usual order is well-ordered.

(3) for every $a \in \Omega$, the subset $H(a) := \{b \in \Omega \mid b < a\}$ is countable.

Let $R' := \Omega \times [0,1)$ with the ordering $(a,x) < (b,y)$ if $a < b$, or $a = b$ and $x \leq y$. Let 0_Ω be the minimum of Ω so that $O = (0_\Omega, 0)$ is the minimum of R'. An open interval in R' has the form $I(\alpha,\beta) := \{\phi \in R' \mid \alpha < \phi < \beta\}$ for $\alpha,\beta \in R'$ or $J(\beta) = \{O\} \cup I(O,\beta) = \{\phi \in R' \mid \phi < \beta\}$. Find an uncountable collection of open intervals such that no intervals intersect. Why is R' not Lindelöf?
R' is called the 'long ray' (R is called a ray, or half-line).

If you are interested in these strange topological spaces, the famous reference is Steen and Seebach's Counterexamples in Topology. An online reference is the database website π-Base `https://topology.jdabbs.com/`.

Terminology
Bild = image.
Urbild = pre-image.
Homöomorphismus = homeomorphism.
Umgebung = neighbourhood.
unabzählbar = uncountable.
zusammenhängend = connected.