Introduction to Partial Differential Equations
Revision Tutorial

Martin Schmidt – Ross Ogilvie

Universität Mannheim

HSS 2021

How to use this Revision Tutorial

- What is examinable is the script with a focus on proofs.
- This is a study aid, not a study substitute.
- Each section tries to highlight a common theme.
- Not comprehensive, not strictly ordered.
- References eg S1.1, Ex1.

What is a PDE?

What sort of functions are allowed to be solutions?

Include boundary values

- What is a PDE?
- Three main questions: Regularity, existence, and uniqueness
- S2.3, S2.5

Consider $y' = y \Rightarrow y(x) = Ae^x$.

we see $y(0) = A \Longleftrightarrow \{Ae^x\}$

$y'(0) = A \Longleftrightarrow \{Ae^x\}$

Classifying PDEs

- Order. — The highest derivative that occurs.
- Linearity and Homogeneity.

$\underbrace{(\partial_t - \partial_x)}_{L} u = 0$ $\quad L(au+bv) = a\,Lu + b\,Lv$ $\quad L(u+v) = Lu + Lv = 0 + f = f.$

$Lu = 0$ homogeneous $\quad Lv = f$ inhom.

- Elliptic, Parabolic, Hyperbolic Ex15, Ex24

$\sum_{ij} \boxed{a_{ij}}\, \partial_i \partial_j u = a_{11}\partial_1^2 u + a_{12}\partial_1\partial_2 u + a_{21}\partial_2\partial_1 u + a_{22}\partial_2^2 u$

symmetric

$+ \sum b_i \partial_i u + cu = f$

- Exemplars S2.2.

• all eigenvalues same sign → elliptic
• all eigenvalues same sign, except one is zero → parabolic
• only one eigenvalue opposite sign to all other → hyperbolic

E: Laplace's Eqn $-\Delta u = 0$ $\quad \Delta u = \sum \frac{\partial^2 u}{\partial x_i^2}$

P: Heat Eqn $\partial_t u - \Delta u = 0$ $\quad \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$

H: Wave Eqn $\partial_t^2 u - \Delta u = 0$ $\quad \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$

Domains and boundary conditions - S2.6

- Typical Domains open , sometimes bounded, boundary is a submanifold

Often look balls $B(x,r)$,

$$\Omega_T = \Omega \times (0,T]$$

$$\partial_p \Omega_T = \sqcup = \partial\Omega \times (0,T] \cup \Omega \times \{0\}$$

- Dirichlet, Neumann, and Cauchy Problems.
- Well-posedness (Ex31).

↳ given the boundary values does there exist the possibility of soln?

D: give on boundary value of u

N: give on boundary value of derivatives

C1: Both - wave

C2: initial condition + boundary values - Heat.

Chain Rule - Ex2, Ex20

1. $\frac{\partial}{\partial\theta}\Big(u(r\cos\theta, r\sin\theta)\Big)$
2. $\frac{\partial^2}{\partial t^2}\Big(F(x-t^2)\Big)$

matrix multiplication

linear op = matrix of partial derivatives

$$D(f\circ g)(x) = Df(g(x)) \circ \overbrace{Dg}(x)$$

$$Df = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f^m}{\partial x_1} & \cdots & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \nabla f^1 \\ \hline \nabla f^2 \\ \hline \vdots \\ \hline \nabla f^m \end{pmatrix}$$

$$(x,y) = \Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\frac{\partial}{\partial\theta}(u\circ\Phi)$$

$$D(u\circ\Phi) = \left(\frac{\partial(u\circ\Phi)}{\partial r} \quad \frac{\partial(u\circ\Phi)}{\partial\theta}\right) = (Du)(\Phi(r,\theta)) \cdot D\Phi = \nabla u \cdot \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{pmatrix}$$

$$= \nabla u \cdot \left(\frac{\partial\vec{x}}{\partial r} \,\middle|\, \frac{\partial\vec{x}}{\partial\theta}\right) = \left(\nabla u\cdot\frac{\partial\vec{x}}{\partial r} \,\middle|\, \nabla u\cdot\frac{\partial\vec{x}}{\partial\theta}\right)$$

$$\frac{\partial u\circ\Phi}{\partial y_i} = \nabla u \cdot \frac{\partial\vec{x}}{\partial y_i}$$

1. $$\frac{\partial}{\partial\theta}\big(u(r\cos\theta, r\sin\theta)\big) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} \quad *$$

$$= -r\sin\theta \boxed{\frac{\partial u}{\partial x}} + r\cos\theta\frac{\partial u}{\partial y}.$$

$$\frac{\partial u}{\partial x}\circ\Phi$$

2. $$\frac{\partial^2}{\partial t^2}F(x-t^2) \qquad v = x-t^2$$

$$\frac{\partial}{\partial t}F(x-t^2) = \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} = F'(x-t^2)\cdot(-2t)$$

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}F(x-t^2)\right] = -2F'(x-t^2) + F''(x-t^2)(-2t)^2$$

Submanifold and Integrals - S2.1

- If $\Phi : U \subset \mathbb{R}^k \to O$ (Definition 2.1) the integral on O is defined (Definition 2.3) to be

$$\int_O f \, d\sigma = \int_U f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu.$$

- Eg $O = \{x^2 + y^2 = 1, y > 0\}$ and $f = x$. Ex11
- Partition of Unity (Definition 2.3).

$S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2$

$\Phi_1 : \overset{\theta \in}{(0, 2\pi)} \longmapsto (\cos\theta, \sin\theta)$

U_1

$O_1 = S^1 \setminus \{(1,0)\}$

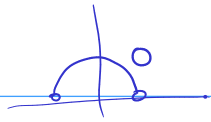
$\downarrow$

O_2

$\Phi_2 : \theta \in (-\pi, \pi) \longmapsto (\cos\theta, \sin\theta)$

U_2

$$\Phi : \theta \in U = (0,\pi) \mapsto (\overset{x}{\cos\theta}, \overset{y}{\sin\theta})$$



$$\int_O x \, d\sigma = \int_U \cos\theta \sqrt{\det\left((\Phi')^T \Phi'\right)} \, d\theta = \int_0^{\pi} \cos\theta \, d\theta = 0.$$

$$\Phi' = D\Phi = \begin{pmatrix} \frac{\partial \Phi^1}{\partial \theta} \\ \frac{\partial \Phi^2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$(\Phi')^T \Phi' = (-\sin\theta \, , \cos\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \left(\sin^2\theta + \cos^2\theta \right) = (1)$$

$\det = 1$

A partition of unity subordinate to a cover O_i $(\cup O_i = A)$ is a collection of smooth functions $h_i : A \longrightarrow [0,1]$

- $h_i = 0$ outside O_i
- at every point of x $\quad \sum h_i(x) = 1$.

$f = 1 \cdot f = \sum f h_i \qquad f h_i$ is zero outside of O_i

$$\int_A f \, d\sigma = \int_A \sum f h_i \, d\sigma = \sum \int_{O_i} f h_i \, d\sigma$$

Divergence Theorem 2.5

- Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ with outward point normal N. Let $F : \bar{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously extends to $\partial\Omega$. Then we have

$$\int_\Omega \nabla \cdot F \, d\mu = \int_{\partial\Omega} F \cdot N \, d\sigma.$$

  - Ex11(e), 12

- Integration by Parts Suppose F is zero outside $[-R,R]^n$

$F = \begin{bmatrix} 0 \\ \vdots \\ fg \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th}$ $\quad \nabla \cdot F = \sum \frac{\partial F^i}{\partial x_i} = \partial_i(fg)$

$\int_\Omega \partial_i(fg) = \int_{\partial\Omega} F \cdot N \, d\sigma = 0$

$\int_\Omega (\partial_i f) g = -\int_\Omega f(\partial_i g).$

Properties of means - Ex19, Ex22, Ex23a

- Means, or weighted averages, are $M(u, x, r) = (C_r)^{-1} \int_{x+A_r} u(y)w(y-x)$. $w \geq 0$
- $x + A_r$ is a set 'centred' at x with 'radius' r and $C_r = \int_{A_r} w(x)$ is the normalisation.
- Spherical mean (Laplace Equation, Wave equation): set is a sphere $\partial B(x, r)$, weight is 1, $C_r = n\omega_n r^{n-1}$.
- Heat mean: set is a heat ball $E(x, t, r)$, weight $w(x, t) = |x|^2/t^2$.
- The average of a constant is the constant $M(c, x, r) = c$.

$M(1,x,r) = (C_r)^{-1} \int_{x+A_r} 1\, w(y-x)\, dy = (C_r)^{-1} \int_{A_r} w(z) dz = \frac{C_r}{C_r} = 1.$

- For continuous functions $\lim_{r\to 0^+} M(u, x, r) = u(x)$. $A_r \to 0$

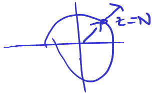
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in B(x,\delta)\quad f(x)-\varepsilon < f(y) < f(x)+\varepsilon$

$M(f(x)-\varepsilon, x, r) \leq M(f, x, r) \leq M(f(x)+\varepsilon, x, r)$ $\forall r$ s.t. $A_r \subset B(x,\delta)$

constants in y

$f(x)-\varepsilon \leq M(f,x,r) \leq f(x)+\varepsilon$

Mean value property - S3.2, S4.3, S5.2



- What is $\partial_r M$? $\partial_r M = 0$
- Proof of Mean Value Property 3.3: $u \in C^2$ $\Delta u = 0$.
  $\frac{\partial}{\partial r} \frac{1}{n\omega_n} \int_{\underline{\partial B(0,1)}} u(x + rz)\, d\sigma(z) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \frac{\partial}{\partial r}(u(x+rz))\, d\sigma(z)$

$= \frac{1}{n\omega_n} \int \nabla u \cdot \frac{\partial(x+rz)}{\partial r}\, d\sigma(z) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla u \cdot N\, d\sigma(z)$

$= \frac{1}{n\omega_n} \int_{B(0,1)} \nabla \cdot (\nabla u) = \frac{1}{n\omega_n} \int \Delta u = 0.$

- Harmonic functions are equal to their spherical means (of any radius). Ditto heat functions. $u(x) = M(u, x, r)$ $\forall r$, $\partial B(x,r) \subseteq \Omega$.
- Spherical means of Wave Equation obey Euler-Poisson-Darboux equation (Lemma 5.2).

Maximum principles - S3.3, S4.4

- For elliptic and parabolic, non-degenerate critical points cannot be extrema. Ex24
- Local Maximum Principle: If u has a maximum at x, then it is constant on $B(x,r) \subset \Omega$. then is is constant on $E(x,t,r) \subset \Omega_T$.

Laplace

Heat

Near x: $v(y) = u(x) - u(y) \geq 0$

$$0 = v(x) = M(v,x,r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y)\,dy \Rightarrow v(y) = 0 \Leftrightarrow u(x) = u(y)$$

- Strong Maximum Principle 3.10: If u has a maximum on an open, path-connected set Ω or Ω_T, then it is constant.

Ω_T

- Weak Maximum Principle 3.11: On a bounded domain, the maximum is taken on the boundary.
- Weak Maximum Principle gives uniqueness for Dirichlet problem.

$\Delta u = 0$

$u = g$ on $\partial\Omega$

$v = u_1 - u_2 \Rightarrow \Delta v = 0$

$v = g - g = 0$ on $\partial\Omega$

$\Delta(-v) = 0$

$-v = 0$ on $\partial\Omega$

$\max(-v) = 0$

$-\min v = 0$

$u_1 = u_2 \Leftarrow \max v = 0.$

Subharmonic and Inequality of Solutions - Thm 3.13, Ex25, Ex26, Ex35

- In proof of Mean Value Property, we used $\Delta u = 0$. For subsolutions we get that u is less than its mean and maximum principle.
- Instead of uniqueness of Dirichlet problem, get inequality of solutions.

$Lu \leq 0$

Subharmonic $-\Delta u \leq 0$ $\quad \partial_r M \geq 0 \Rightarrow u(x) \leq M(u,x,r)$

Consider two sol^n $\partial_t u_i - \Delta u_i = f_i$

for $\partial\Omega$ $\quad u_i = g_i$

for $t=0$ $\quad u_i = h_i$

$f_1 \leq f_2$
$g_1 \leq g_2$
$h_1 \leq h_2$

$\xrightarrow{V = u_1 - u_2}$

$\partial_t V - \Delta V = f_1 - f_2 \leq 0$ *

on $\partial\Omega$ $\quad V = g_1 - g_2 \leq 0$

on $t=0$ $\quad V = h_1 - h_2 \leq 0$.

Because V is a subsol^n

max V is on $\partial_p \Omega_T$

$\max V \leq 0$

$V \leq 0$

$\longleftarrow$

$u_1 \leq u_2$ on all $\overline{\Omega_T}$

Dirichlet's principle for harmonic - S3.5

- Alternative method to prove uniqueness.
- Functional $I_{f,g} : \{w \in C^2(\overline{\Omega}) \mid w|_{\partial\Omega} = g\} \to \mathbb{R}$ given by $I_{f,g}(w) = \int_\Omega 0.5\|\nabla w\|^2 - wf$. $-\Delta w = f$
- Minimiser is a solution to Laplace equation Thm 3.25.
- Difference of two harmonic functions minimises $I_{0,0}$, implies uniqueness.
- There's a short calculation for the heat equation at end S4.4 with $e(t) = \int_\Omega |u|^2 dx$, $f = g = 0$, Ω does not need to be bounded. It shows $\partial_t e \leq 0$.

$v = u_1 - u_2$. $\Delta v = 0 = f$, $v = 0 = g$ on bondary. $I_{0,0}$ has 0 as a minimum.

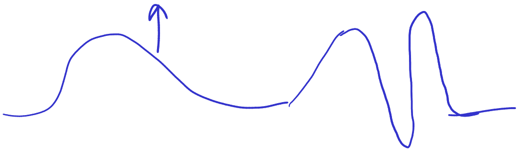
Energy of a Wave - S5.8

- Theorem 5.7: Inhomogeneous wave equation with initial and boundary conditions, Ω bounded domain. Then solution is unique.
- $E(t) = \frac{1}{2}\int_\Omega \overset{\text{kinetic}}{\underline{(\partial_t u)^2}} + \overset{\text{potential}}{\underline{\|\nabla u\|^2}}\, dx$. E is constant over time.
- The only solution with zero on the boundary is zero.

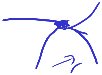
$$\partial_t E = \frac{1}{2}\int_\Omega 2(\partial_t u)(\partial_t^2 u) + 2\sum_{i=1}^{n}(\partial_i u)(\partial_t \partial_i u)$$

$$= \frac{1}{2}\int_\Omega (2\partial_t u)\sum \partial_i^2 u + 2\sum(\partial_i u)(\partial_i \partial_t u)$$

$$= \frac{1}{2}\int_\Omega (2\partial_t u)\sum \partial_i^2 u - 2\sum(\partial_i^2 u)(\partial_t u) = 0$$



Fundamental Solutions - S3.1, S4.1



- Laplace Eqn: The Laplacian has many symmetries (Ex20) so we seek radially symmetric solutions.
- Due to Ex13, integral on every ball enclosing $x = 0$ the same. Choose constants to make this 1 and vanishing at infinity:

$$\Phi_L(x) = \begin{cases} -\frac{1}{2\omega_2} \ln |x| & \text{for } n = 2, \\ \frac{1}{n(n-2)\omega_n} |x|^{-(n-2)} & \text{for } n > 2. \end{cases}$$

- Heat Eqn: Characteristics of the form $t^{-1}|x|^2$. Choose constants so it vanishes at infinity and $\int_{\mathbb{R}^n} \Phi \, dx = 1$ (Lemma 4.2). Extend to $t \leq 0$ by zero.

$$\Phi_H(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp -\frac{|x|^2}{4t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, (x,t) \neq (0,0). \end{cases}$$

Solving Inhomogeneous Equations - S3.1, S4.2

- As distributions, fundamental solutions obey $\boxed{L\Phi = \delta}$. Theorem 3.2 and Theorem 4.4, Ex34.
- Gives a solution of inhomogeneous problem on $\mathbb{R}^n$.

$$\partial(f*g) = \partial f * g$$

$$L(\underbrace{\Phi * f}_{u}) = (L\Phi) * f = \delta * f = f.$$

$$Lu = f.$$

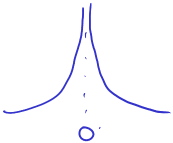
$$-\Delta u = f$$

$$(\partial_t - \Delta)u = f$$

- Proof typically splits integral into part near singularity and part away, eg I_ϵ, J_ϵ and u_ϵ.

$-\Delta u = f$ electric charge

$(\partial_t - \Delta)\underset{}{\overset{\text{temp}}{u}} = \overset{\text{heatsource.}}{f}$



Green's Functions and Heat Kernel - S3.4, S4.5

- Generalisation of Fundamental Solution to other domains $\Omega \subset \mathbb{R}^n$.
- Defn 3.18: Green's function $G_\Omega : \{(x, y) \in \Omega \times \Omega \mid x \neq y\} \to \mathbb{R}$ obeys for all $x \in \Omega$:
  - i. $y \mapsto G_\Omega(x, y) - \Phi(x - y)$ is harmonic.
  - ii. $y \mapsto G_\Omega(x, y)$ extends to the boundary continuously and is zero.
- Defn 4.14: Heat Kernel $H_\Omega : \{(x, y) \in \Omega \times \Omega \mid x \neq y\} \times \mathbb{R}^+ \to \mathbb{R}$ obeys for all $(x, t) \in \Omega \times \mathbb{R}^+$:
  - i. $y \mapsto H_\Omega(x, y, t) - \Phi(x - y, t)$ solves the heat equation with initial condition zero.
  - ii. $y \mapsto G_\Omega(x, y)$ extends to the boundary continuously and is zero.
- Not all domains have a Green's function. Ex31. $B(0,1)\setminus\{0\}$.
- Green's functions are symmetric Thm 3.19 and for bounded domains unique.

$\tilde{G} = G_1 - G_2 = G_1 - \Phi(x-y) - (G_2 - \Phi(x-y))$ = harmonic function in y.

$\tilde{G}$ vanish on $y \in \partial\Omega$. $\tilde{G} = 0$.

Representation Formula - S3.4, S4.5 Ex29, Ex39

- Green's Representation Theorem 3.16: For an open and bounded domain Ω to which the divergence theorem applies and $u \in C^2(\overline{\Omega})$:

$$u(x) = -\int_\Omega G_\Omega(x,y)\overset{f(y)}{\cancel{\triangle_y u(y)}}\, d^n y - \int_{\partial\Omega} \overset{g(z)}{\cancel{u(z)}}\nabla_z G_\Omega(x,z)\cdot N\, d\sigma(z).$$

- Theorem 4.16

$$\begin{aligned} u(x,t) = &\int_0^t \int_\Omega \overbrace{(\dot{u}(y,s) - \triangle u(y,s))}^{f(y,s)} H_\Omega(x,y,t-s)\, d^n y\, ds \\ &- \int_0^t \int_{\partial\Omega} \overbrace{u(z,s)}^{g(z,s)} \nabla_z H_\Omega(x,z,t-s)\cdot N(z)\, d\sigma(z)\, ds \\ &+ \int_\Omega \underbrace{u(y,0)}_{h(y)} H_\Omega(x,y,t)\, d^n y. \end{aligned}$$

- Proves existence of Dirichlet problems constructively.

Heat equation in $\mathbb{S}^1$ - S4.7, Ex40

$-\Delta f_k = \lambda_k f_k$
Then $e^{-\lambda_k t} f_k$ solves the heat eqn.

- This section gives us an alternate method to construct heat kernels. All functions can be written as the sum (or integral) of eigenfunctions of the Laplacian.
- If the initial condition is an eigenfunction f_k of $-\Delta$ with eigenvalue λ_k a solution is $e^{-\lambda_k t} f_k(x)$. Ex32 separable solutions.
- Writing $h(x) = \int \hat{h}(k) f_k(x)\, dk$ gives the solution

$$u(x,t) = \int \hat{h}(k) \underline{e^{-\lambda_k t} f_k(x)}\, dk.$$

- If have a periodic initial condition, only periodic eigenfunctions are needed, we get the heat kernel on S^1

heat kernel of S^1

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{h}(k) e^{-\lambda_k t} f_k(x) = \int \left[\sum_{k\in\mathbb{Z}} e^{-2\pi i k y} e^{-\lambda_k t} f_k(x) \right] h(y)\, dy$$

- To handle $\underline{[0,1]}$: again use eigenfunctions, or reflect S^1.

Consider $L = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$v = \begin{pmatrix} a \\ b \end{pmatrix} = a v_1 + b v_2$$

$$Lv = a L v_1 + b L v_2 = a v_1 + 2 b v_2$$

$v_1 \perp v_2$

$v = \sum \langle v, v_i \rangle v_i$ if v_i are orthonormal.

$h(x) = \int \hat{h}(k) f_k(x)\, dk$ $\quad \hat{h}$ the Fourier transform of f.

S^1
- a function on $[0,1]$ with $f(0) = f(1)$ $\iff$
- a function on $\mathbb{R}$ with $f(x) = f(x+1)$

Eigenfunctions $f_k(x) = e^{2\pi i k \cdot x}$ are periodic $k \in \mathbb{Z}^n$

If h is periodic then it is a sum of periodic eigenfunctions

f is a function on $[0,1]$

$g = \begin{cases} f(x) & x \in [0,1] \\ f(2-x) & x \in [1,2] \end{cases}$ $\quad [0,2]$ $\quad g(2) = f(2-2) = f(0) = g(0)$

Transport Equation and D'Alembert's Formula - S1.1, S1.2, S5.1, Ex 41

$\sum b_i \partial_{x_i} u$

$\eta = x - bt \quad \zeta = x + bt$

$x = \frac{1}{2}(\zeta + \eta) \quad t = \frac{1}{2b}(\zeta - \eta)$

- The Transport equation: $(\partial_t + \overbrace{b \cdot \nabla})u = 0$.

$\partial_\zeta = \frac{\partial x}{\partial \zeta}\partial_x + \frac{\partial t}{\partial \zeta}\partial_t = \frac{1}{2}\partial x + \frac{1}{2b}\partial_t = \frac{1}{2b}(\partial_t + b\,\partial_x)$

- Solved by $g(x - bt)$ for initial condition $u(x, 0) = g(x)$.

$\partial_t(g(x-bt)) = \nabla g \cdot \frac{\partial(x-bt)}{\partial t} = \nabla g \cdot (-b)$

$\partial_\zeta u = 0$

$\nabla(g(x-bt)) = \nabla g \; D_x(x-bt) = \nabla g \; I = (\nabla g)(x-bt)$ $+$ $= 0$.

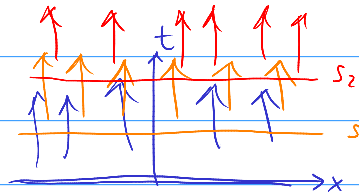
- 1D Wave Equation factors into two transport equations $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$

$(\partial_t - \partial_x)v = 0$
$(\partial_t + \partial_x)u = v$

- D'Alembert's Formula: $u(x, t) = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$.
- Duhamel's principle: turn an inhomogeneous problem into an initial value one.

$\partial_t u + b\partial_x u = f(x,t)$

For every parameter $s>0$, we solve the following for v_s



$\partial_t v_s + b\partial_x v_s = 0$ (b)

$v_s(x,s) = f(x,s)$ (a)

Idea: $u(x,t) = \int_0^t v_s(x,t)\, ds$

$\partial_x u = \int_0^t \partial_x v_s\, ds$ $\qquad$ $\partial_t u = v_s(x,t)\Big|_{s=t} + \int_0^t \partial_t v_s\, ds.$

$\partial_t u + b\partial_x u = \underbrace{v_t(x,t)}_{(a)\ f(x,t)} + \int_0^t \underbrace{\partial_t v_s + b\partial_x v_s}_{(b)\ 0}\, ds = f(x,t)$

What is v_s? Solve $\partial_t v_s + b\partial_x v_s = 0$

$v_s(x,s) = f(x,s)$

Make the translation $\tau = t-s \qquad \partial_t = \partial_\tau$

$\partial_\tau v_s - b\partial_x v_s = 0$

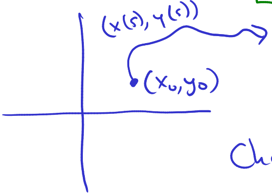
$v_s(x,\tau=0) = f(x,s)$

Solved by $v_s = f(x-b\tau, s) = f(x-b(t-s), s)$

$u(x,t) = \int_0^t f(x-b(t-s), s)\, ds.$

Method of Characteristics - S1.5 Ex8-10

- A generalisation of the transport equation for non-constant coefficients.
- You choose a path along which the values of the function can be described by an ODE system, parametrised by the initial point.
- Example: $\boxed{x}\partial_x u + \boxed{2y}\partial_y u = u$



$$z(s) = u(x(s), y(s))$$

$$\frac{dz}{ds} = \frac{\partial u}{\partial x}\boxed{\frac{dx}{ds}} + \frac{\partial u}{\partial y}\boxed{\frac{dy}{ds}}$$

Choose $\frac{dx}{ds} = x$, $\frac{dy}{ds} = 2y$ then $\frac{dz}{ds} = x\partial_x u + 2y\partial_y u$
$= u = z$

System of ODEs

$$\frac{d}{ds}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2y \\ z \end{pmatrix} \Rightarrow \begin{aligned} x &= \boxed{x_0} e^{s} \\ y &= \boxed{y_0} e^{2s} \\ z &= z_0 e^{s}. \end{aligned}$$

$$z_0 = u(x(0), y(0)) = u(x_0, y_0)$$

Consider this PDE on $\{(x,y) \mid y>0\}$ with boundary conditions $u(x,1) = 2x^2$

We should choose $(x_0, y_0) \in \{(x,1)\}$ ie $y_0 = 1$.

$z_0 = u(x_0, 1) = 2x_0^2$

$$\left.\begin{aligned} x &= x_0 e^s \\ y &= e^{2s} \\ z &= 2x_0^2 e^s \end{aligned}\right\}$$ Try to eliminate x_0, s.

$e^s = \sqrt{y}$ $\qquad x_0 = \frac{x}{e^s} = \frac{x}{\sqrt{y}}$

$z = 2\left(\frac{x}{\sqrt{y}}\right)^2 \sqrt{y} = 2x^2/\sqrt{y}$

$$\boxed{u(x,y) = 2x^2/\sqrt{y}}$$

$x\partial_x u = 4x^2/\sqrt{y}$ $\qquad 2y\partial_y u = 4x^2 \cancel{y} \cdot \left(-\frac{1}{2}\right)\frac{1}{\cancel{y}\sqrt{y}} = -2x^2/\sqrt{y}$

"+" $= 2x^2/\sqrt{y} = u$.

Wave Equation and Method of Descent - S5.3-5.6 Ex44

- 1D Wave Equation on $\mathbb{R}$ can be solved by D'Alembert's formula.
- 1D Wave Equation on $\mathbb{R}^+$ transformed to 1D Wave Equation on $\mathbb{R}$ by reflection principle.
- The spherical means of solutions to the wave equation obey the Euler-Poisson-Darboux equation.
- In odd dimensions, there is a transformation that reduces the EPD equation to the 1D Wave Equation on $\mathbb{R}^+$.
- Any solution to the wave equation extends to a solution in higher dimensions, if you let it be constant in the extra directions: $u \mapsto \bar{u}$.
- In even dimensions, extend the solution to one dimension higher, then solve.
- All these transformations change the PDE, but also the boundary/initial conditions.

Regularity of Harmonic Functions

- Harmonic functions are by definition $u \in C^2(\Omega)$ with $\Delta u = 0$.
- A harmonic distribution is a distribution $U : \mathcal{D}'(\Omega)$ with $\Delta U = 0$ in the sense of distributions. $\forall$ test functions $U(\Delta\varphi) = 0$
- Weak Mean Value Property 3.6, Ex 27: For all balls $B(x, r) \subset \Omega$ and all test functions $\psi : (0, r) \to \mathbb{R}$ with total mass zero $\int \psi = 0$, the distribution is zero for the test function

$$\underline{f_{x,\psi}(y)} = \frac{\psi(|y-x|)}{n\omega_n |y-x|^{n-1}}.$$

- All harmonic distributions have the weak mean value property (Lemma 3.6).
- Weyl's Lemma 3.7: All harmonic distributions come from a smooth harmonic function. $U = F_u$ for $u \in C^\infty$, $\Delta u = 0$

Other Theorems for Harmonic Functions

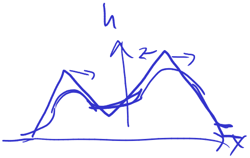
- Analytic Cor3.22: All harmonic functions are analytic. Proof follows from representation formula.
- Liouville's theorem 3.5 Ex23: The only harmonic functions on $\mathbb{R}^n$ that are bounded are the constant functions.
- Removable Singularity Lemma 3.24: If a harmonic function on $\Omega \setminus \{x\}$ is bounded, it extends to a harmonic function on Ω.
- Unique Continuation Ex30: There is at most one harmonic extension of a harmonic function to a larger domain.

Non analytic $\begin{cases} e^{-1/x} & x>0 \\ 0 & x\le 0. \end{cases}$ at $x=0$ $T(x) = 0 + 0x + 0x^2 + \dots$

Consider $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ $|f| \le 1$ no continuous extension to $(0,0)$

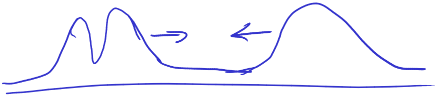
Solutions of the Heat Equation

- Cor 4.26: Any solution of heat equation is smooth in t, analytic in x.
- Ex 36: For open and bounded domains with boundary conditions that are constant in time. If there is a steady state solution, then all other continuous initial conditions tend to the steady state solution as $t \to \infty$.
- Theorem 4.11: For the heat equation on $\mathbb{R}^n$ with continuous bounded initial condition, there is at most one solution with $u(x,t) \le Ae^{a|x|^2}$.

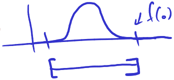


If u heat sol^n with 0 on $\partial\Omega \times \mathbb{R}^+$

$u \to 0$ regardless of initial cond.



Definition of Distributions - S2.4, Ex17



- Support: $\operatorname{supp} f = \text{closure } \{x \mid f(x) \neq 0\}$. The support of a function is compact $\Leftrightarrow$ it is bounded.
- Test functions $\mathcal{D}(\Omega)$: the set $C_0^\infty(\Omega, \mathbb{R})$ of smooth functions with compact support in Ω with a certain topology (a non-norm topology).
- The topology comes from the semi-norms $\|\phi\|_{K,\alpha} = \sup_{x\in K} |\partial^\alpha \phi|$.
- Distribution are linear and continuous functions $F \in \mathcal{D}'(\Omega)$. Continuity means: for all compact $K \subset \Omega$, there exist multiindices α_i and constants C_i such that for all test functions with $\operatorname{supp} \phi \subseteq K$:

$$|F(\phi)| \leq \sum C_i \|\phi\|_{K,\alpha_i}.$$

- For any $f \in L^1_{loc}(\Omega)$ there is a distribution $F_f \in \mathcal{D}'(\Omega)$ given by $F_f(\phi) = \int_\Omega f\phi$. This association is injective Lemma 2.9.

for any K, $\phi \in D(K)$

$$|\delta(\phi)| = |\phi(0)| \leq \sup_K |\phi| =: \|\phi\|_{K,0}.$$

$$|F_f(\phi)| = \left|\int_K f\phi\right| \leq \underbrace{\left[\left(\sup_K |f|\right)\mu(K)\right]}_{C_1} \underbrace{\sup_K |\phi|}_{\|\phi\|_{K,0}}$$

Operations on Distributions

- Distributions are a vector space over $\mathbb{R}$: $(aF + bG)(\phi) = aF(\phi) + bG(\phi)$.
- Differentiation: $\partial_i F$ is the distribution defined by $\phi \mapsto -F(\partial_i \phi)$.

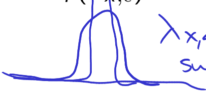
$F_{\partial f}(\phi) = \int_\Omega (\partial f)\phi = -\int_\Omega f(\partial \phi) = -F_f(\partial\phi)$

$F_{fg}(\phi) = \int f(g\phi) = F_f(g\phi)$

- Multiplication with a smooth function g: $(gF)(\phi) = F(g\phi)$.
- Convolution with test function g: $(g * F)(\phi) = F(\phi * \mathrm{P}g)$ where $\mathrm{P}g(x) = g(-x)$.

$F_{f*g}(\phi) = \ldots\ldots = F_f(\phi * P_g)$

- $\delta * F = F$.
- Lemma 2.7: The convolution of a distribution corresponds to a smooth function. Any dist F $\exists h \in C^\infty$ s.t $g * F = F_h$
- Lemma 2.8: for $f \in C(\Omega)$ we can undo the correspondence with $F_f(\lambda_{x,\epsilon}) \to f(x)$ as $\epsilon \to 0$.



$\lambda_{x,\varepsilon} \in C^\infty$

$\mathrm{supp}\, \lambda_{x,\varepsilon} \subset B(x,\varepsilon)$

$\int \lambda_{x,\varepsilon} = 1.$

Weak solutions

- If a function solves a linear PDE, its distribution also solves the PDE (in the sense of distributions).
- Are there other solutions if we look among distributions? This is the most general setting for the PDE.
- Allows you to consider discontinuous boundary conditions.
- You might find that the only distributions that solve the PDE correspond to functions.

Suppose $(\partial_t - \partial_x) u = 0$

$$((\partial_t - \partial_x) F_u)(\varphi) = (\partial_t F_u - \partial_x F_u)(\varphi) = \partial_t F_u(\varphi) - \partial_x F_u(\varphi)$$

$$= -F_u(\partial_t \varphi) + F_u(\partial_x \varphi) = \int_\Omega -u \partial_t \varphi + u \partial_x \varphi$$

$$= \int [(\partial_t u) - (\partial_x u)] \varphi = \int 0 \varphi = 0 .$$

Weak solutions to Transport and 1D wave - Ex18, Ex2.10, Ex42

- We have seen that solutions are $F(x-bt)$ and $\boxed{F(x-t)}+G(x+t)$ respectively when F and G are sufficiently differentiable.
- For all L^1_{loc} function the corresponding distributions are solutions.

Show $f(x-t)$ solves transport eqn in the sense of distributions

$$F(\phi)=\int_{\mathbb{R}^2} f(x-t)\phi(x,t)\,dx\,dt.$$

$$((\partial_t+\partial_x)F)(\phi)=F(-\partial_t\phi-\partial_x\phi)=\int_{\mathbb{R}^2} f(x-t)(-\partial_t\phi-\partial_x\phi)\,dx\,dt.$$

Let $u=x-t \quad v=x+t$ $\qquad dx\,dt=\begin{vmatrix} 1 & -1 \\ 1 & 1\end{vmatrix}=2\,du\,dv$

$$\partial_t\phi=\frac{\partial u}{\partial t}\frac{\partial\phi}{\partial u}+\frac{\partial v}{\partial t}\frac{\partial\phi}{\partial v}=-\partial_u\phi+\partial_v\phi$$

$$\partial_x\phi=\partial_u\phi+\partial_v\phi$$

$$=\int_{\mathbb{R}^2} f(u)\left[\cancel{\partial_u}\phi-\partial_v\phi\ \cancel{-\partial_u}\phi-\partial_v\phi\right]2\,du\,dv \qquad (\partial_t+\partial_x)F=0$$

$$=\int_{\mathbb{R}} f(u)\left(\int_{\mathbb{R}}-4\partial_v\phi\,dv\right)du=\int f(u)\cdot 0\,du=0.$$

Weak solutions to first order systems - S1.4, Ex5-7

- Section 1.4 we look for solutions to scalar conversation PDEs (Section 1.3): $\partial_t u + f'(u)\,\partial_x u = 0$ for $f : \mathbb{R} \to \mathbb{R}$. Particularly Burger's equation $f(u) = \frac{1}{2}u^2$.
- These PDEs are not linear, so distribution methods don't apply nicely.
- By method of characteristics, for some initial conditions no C^1 solution possible.
- Instead we look for solutions that are $C^1(\mathbb{R}^2)$ except for certain curves in the domain. We require that desirable Properties hold 'under the integral sign'.
- Theorem 1.11: $f \in C^2$ strictly convex, initial condition is bounded and L^1, then there is a unique solution of the scalar conservation PDE obeying Rakine-Hugonoit and Lax entropy conditions.