Introduction to Partial Differential Equations Revision Tutorial

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HSS 2021

How to use this Revision Tutorial

- What is examinable is the script with a focus on proofs.
- This is a study aid, not a study substitute.
- Each section tries to highlight a common theme.
- Not comprehensive, not strictly ordered.
- References eg S1.1, Ex1.

Basic Notions

Mean Value Properties and Maximum Principles

Energy Methods

Methods of Solution

Behaviour of Solutions

Distributions and Weak Solutions

What is a PDE?

- What is a PDE?
- Three main questions: Regularity, existence, and uniqueness
- S2.3, S2.5

Classifying PDEs

- Order.
- Linearity and Homogeneity.
- Elliptic, Parabolic, Hyperbolic Ex15, Ex24
- Exemplars S2.2.

Domains and boundary conditions - S2.6

- Typical Domains

- Dirichlet, Neumann, and Cauchy Problems.
- Well-posedness (Ex31).

Chain Rule - Ex2, Ex20

1. $\frac{\partial}{\partial\theta}\Big(u(r\cos\theta, r\sin\theta)\Big)$
2. $\frac{\partial^2}{\partial t^2}\Big(F(x - t^2)\Big)$

Submanifold and Integrals - S2.1

- If $\Phi : U \subset \mathbb{R}^k \to O$ (Definition 2.1) the integral on O is defined (Definition 2.3) to be

$$\int_O f \; d\sigma = \int_U f \circ \Phi \; \sqrt{\det((\Phi')^T \Phi')} d\mu.$$

- Eg $O = \{x^2 + y^2 = 1, y > 0\}$ and $f = x$. Ex11
- Partition of Unity (Definition 2.3).

Divergence Theorem 2.5

- Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ with outward point normal N. Let $F : \bar{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously extends to $\partial\Omega$. Then we have

$$\int_\Omega \nabla \cdot F \, d\mu = \int_{\partial\Omega} F \cdot N \, d\sigma.$$

  - Ex11(e), 12
- Integration by Parts

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Properties of means - Ex19, Ex22, Ex23a

- Means, or weighted averages, are $M(u, x, r) = (C_r)^{-1} \int_{x+A_r} u(y) w(y - x)$.
- $x + A_r$ is a set 'centred' at x with 'radius' r and $C_r = \int_{A_r} w(x)$ is the normalisation.
- Spherical mean (Laplace Equation, Wave equation): set is a sphere $\partial B(x, r)$, weight is 1, $C_r = n\omega_n r^{n-1}$.
- Heat mean: set is a heat ball $E(x, t, r)$, weight $w(x, t) = |x|^2/t^2$.
- The average of a constant is the constant $M(c, x, r) = c$.
- For continuous functions $\lim_{r\to 0^+} M(u, x, r) = u(x)$.

Mean value property - S3.2, S4.3, S5.2

- What is $\partial_r M$?
- Proof of Mean Value Property 3.3:
  $\frac{\partial}{\partial r} \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x + rz)\, d\sigma(z)$
- Harmonic functions are equal to their spherical means (of any radius). Ditto heat functions.
- Spherical means of Wave Equation obey Euler-Poisson-Darboux equation (Lemma 5.2).

Maximum principles - S3.3, S4.4

- For elliptic and parabolic, non-degenerate critical points cannot be extrema. Ex24
- Local Maximum Principle: If u has a maximum at x, then it is constant on $B(x, r) \subset \Omega$. then is is constant on $E(x, t, r) \subset \Omega_T$.

- Strong Maximum Principle 3.10: If u has a maximum on an open, path-connected set Ω or Ω_T, then it is constant.
- Weak Maximum Principle 3.11: On a bounded domain, the maximum is taken on the boundary.
- Weak Maximum Principle gives uniqueness for Dirichlet problem.

Subharmonic and Inequality of Solutions - Thm 3.13, Ex25, Ex26, Ex35

- In proof of Mean Value Property, we used $\Delta u = 0$. For subsolutions we get that u is less than its mean and maximum principle.
- Instead of uniqueness of Dirichlet problem, get inequality of solutions.

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Dirichlet's principle for harmonic - S3.5

- Alternative method to prove uniqueness.
- Functional $I_{f,g} : \{w \in \overline{\Omega} \mid w|_{\partial\Omega} = g\} \to \mathbb{R}$ given by $I_{f,g}(w) = \int_\Omega 0.5\|\nabla w\|^2 - wf$.
- Minimiser is a solution to Laplace equation Thm 3.25.
- Difference of two harmonic functions minimises $I_{0,0}$, implies uniqueness.
- There's a short calculation for the heat equation at end S4.4 with $e(t) = \int_\Omega |u|^2 dx$, $f = g = 0$, Ω does not need to be bounded. It shows $\partial_t e \leq 0$.

Energy of a Wave - S5.8

- Theorem 5.7: Inhomogeneous wave equation with initial and boundary conditions, Ω bounded domain. Then solution is unique.
- $E(t) = \frac{1}{2}\int_{\Omega}(\partial_t u)^2 + \|\nabla u\|^2 \, dx$. E is constant over time.
- The only solution with zero on the boundary is zero.

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Fundamental Solutions - S3.1, S4.1

- Laplace Eqn: The Laplacian has many symmetries (Ex20) so we seek radially symmetric solutions.
- Due to Ex13, integral on every ball enclosing $x = 0$ the same. Choose constants to make this 1 and vanishing at infinity:

$$\Phi_L(x) = \begin{cases} -\frac{1}{2\omega_2} \ln |x| & \text{for } n = 2, \\ \frac{1}{n(n-2)\omega_n} |x|^{-(n-2)} & \text{for } n > 2. \end{cases}$$

- Heat Eqn: Characteristics of the form $t^{-1}|x|^2$. Choose constants so it vanishes at infinity and $\int_{\mathbb{R}^n} \Phi \, dx = 1$ (Lemma 4.2). Extend to $t \leq 0$ by zero.

$$\Phi_H(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp -\frac{|x|^2}{4t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, (x,t) \neq (0,0). \end{cases}$$

Solving Inhomogeneous Equations - S3.1, S4.2

- As distributions, fundamental solutions obey $L\Phi = \delta$. Theorem 3.2 and Theorem 4.4, Ex34.
- Gives a solution of inhomogeneous problem on $\mathbb{R}^n$.
- Proof typically splits integral into part near singularity and part away, eg I_ϵ, J_ϵ and u_ϵ.

Green's Functions and Heat Kernel - S3.4, S4.5

- Generalisation of Fundamental Solution to other domains $\Omega \subset \mathbb{R}^n$.
- Defn 3.18: Green's function $G_\Omega : \{(x, y) \in \Omega \times \Omega \mid x \neq y\} \to \mathbb{R}$ obeys for all $x \in \Omega$:
  i. $y \mapsto G_\Omega(x, y) - \Phi(x - y)$ is harmonic.
  ii. $y \mapsto G_\Omega(x, y)$ extends to the boundary continuously and is zero.
- Defn 4.14: Heat Kernel $H_\Omega : \{(x, y) \in \Omega \times \Omega \mid x \neq y\} \times \mathbb{R}^+ \to \mathbb{R}$ obeys for all $(x, t) \in \Omega \times \mathbb{R}^+$:
  i. $y \mapsto H_\Omega(x, y, t) - \Phi(x - y, t)$ solves the heat equation with initial condition zero.
  ii. $y \mapsto G_\Omega(x, y)$ extends to the boundary continuously and is zero.
- Not all domains have a Green's function. Ex31.
- Green's functions are symmetric Thm 3.19 and for bounded domains unique.

Representation Formula - S3.4, S4.5 Ex29, Ex39

- Green's Representation Theorem 3.16: For an open and bounded domain Ω to which the divergence theorem applies and $u \in C^2(\overline{\Omega})$:

$$u(x) = -\int_{\Omega} G_{\Omega}(x,y) \triangle_y u(y)\, d^n y - \int_{\partial\Omega} u(z) \nabla_z G_{\Omega}(x,z) \cdot N \, d\sigma(z).$$

- Theorem 4.16

$$\begin{aligned} u(x,t) = & \int_0^t \int_{\Omega} (\dot{u}(y,s) - \triangle u(y,s)) H_{\Omega}(x,y,t-s)\, d^n y\, ds \\ & - \int_0^t \int_{\partial\Omega} u(z,s) \nabla_z H_{\Omega}(x,z,t-s) \cdot N(z)\, d\sigma(z)\, ds \\ & + \int_{\Omega} u(y,0) H_{\Omega}(x,y,t)\, d^n y. \end{aligned}$$

- Proves existence of Dirichlet problems constructively.

Heat equation in $\mathbb{S}^1$ - S4.7, Ex40

- This section gives us an alternate method to construct heat kernels. All functions can be written as the sum (or integral) of eigenfunctions of the Laplacian.
- If the initial condition is an eigenfunction f_k of $-\Delta$ with eigenvalue λ_k a solution is $e^{-\lambda_k t} f_k(x)$. Ex32 separable solutions.
- Writing $h(x) = \int \hat{h}(k) f_k(x)\, dk$ gives the solution

$$u(x,t) = \int \hat{h}(k) e^{-\lambda_k t} f_k(x)\, dk.$$

- If have a periodic initial condition, only periodic eigenfunctions are needed, we get the heat kernel on S^1

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{h}(k) e^{-\lambda_k t} f_k(x) = \int \left[\sum_{k\in\mathbb{Z}} e^{-2\pi i k y} e^{-\lambda_k t} f_k(x) \right] h(y)\, dy$$

- To handle $[0,1]$: again use eigenfunctions, or reflect S^1.

Transport Equation and D'Alembert's Formula - S1.1, S1.2, S5.1, Ex 41

- The Transport equation: $(\partial_t + b \cdot \nabla)u = 0$.
- Solved by $g(x - bt)$ for initial condition $u(x, 0) = g(x)$.

- 1D Wave Equation factors into two transport equations $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$.
- D'Alembert's Formula: $u(x, t) = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$.
- Duhamel's principle: turn an inhomogeneous problem into an initial value one.

Method of Characteristics - S1.5 Ex8-10

- A generalisation of the transport equation for non-constant coefficients.
- You choose a path along which the values of the function can be described by an ODE system, parametrised by the initial point.
- Example: $x\partial_x u + 2y\partial_y u = u$

Wave Equation and Method of Descent - S5.3-5.6 Ex44

- 1D Wave Equation on $\mathbb{R}$ can be solved by D'Alembert's formula.
- 1D Wave Equation on $\mathbb{R}^+$ transformed to 1D Wave Equation on $\mathbb{R}$ by reflection principle.
- The spherical means of solutions to the wave equation obey the Euler-Poisson-Darboux equation.
- In odd dimensions, there is a transformation that reduces the EPD equation to the 1D Wave Equation on $\mathbb{R}^+$.
- Any solution to the wave equation extends to a solution in higher dimensions, if you let it be constant in the extra directions: $u \mapsto \bar{u}$.
- In even dimensions, extend the solution to one dimension higher, then solve.
- All these transformations change the PDE, but also the boundary/initial conditions.

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Regularity of Harmonic Functions

- Harmonic functions are by definition $u \in C^2(\Omega)$ with $\Delta u = 0$.
- A harmonic distribution is a distribution $U : \mathcal{D}'(\Omega)$ with $\Delta U = 0$ in the sense of distributions.
- Weak Mean Value Property 3.6, Ex 27: For all balls $B(x, r) \subset \Omega$ and all test functions $\psi : (0, r) \to \mathbb{R}$ with total mass zero $\int \psi = 0$, the distribution is zero for the test function

$$f_{x,\psi}(y) = \frac{\psi(|y - x|)}{n\omega_n |y - x|^{n-1}}.$$

- All harmonic distributions have the weak mean value property (Lemma 3.6).
- Weyl's Lemma 3.7: All harmonic distributions come from a smooth harmonic function.

Other Theorems for Harmonic Functions

- Analytic Cor3.22: All harmonic functions are analytic. Proof follows from representation formula.
- Liouville's theorem 3.5 Ex23: The only harmonic functions on $\mathbb{R}^n$ that are bounded are the constant functions.
- Removable Singularity Lemma 3.24: If a harmonic function on $\Omega \setminus \{x\}$ is bounded, it extends to a harmonic function on Ω.
- Unique Continuation Ex30: There is at most one harmonic extension of a harmonic function to a larger domain.

Solutions of the Heat Equation

- Cor 4.26: Any solution of heat equation is smooth in t, analytic in x.
- Ex 36: For open and bounded domains with boundary conditions that are constant in time. If there is a steady state solution, then all other continuous initial conditions tend to the steady state solution as $t \to \infty$.
- Theorem 4.11: For the heat equation on $\mathbb{R}^n$ with continuous bounded initial condition, there is at most one solution with $u(x, t) \leq Ae^{a|x|^2}$.

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Definition of Distributions - S2.4, Ex17

- Support: $\operatorname{supp} f =$ closure $\{x \mid f(x) = 0\}$. The support of a function is compact $\Leftrightarrow$ it is bounded.
- Test functions $\mathcal{D}(\Omega)$: the set $C_0^\infty(\Omega, \mathbb{R})$ of smooth functions with compact support in Ω with a certain topology (a non-norm topology).
- The topology comes from the semi-norms $\|\phi\|_{K,\alpha} = \sup_{x \in K} |\partial^\alpha \phi|$.
- Distribution are linear and continuous functions $F \in \mathcal{D}'(\Omega)$. Continuity means: for all compact $K \subset \Omega$, there exist multiindices α_i and constants C_i such that for all test functions with $\operatorname{supp} \phi \subseteq K$:

$$|F(\phi)| \leq \sum C_i \|\phi\|_{K,\alpha_i}.$$

- For any $f \in L^1_{loc}(\Omega)$ there is a distribution $F_f \in \mathcal{D}'(\Omega)$ given by $F_f(\phi) = \int_\Omega f \phi$. This association is injective Lemma 2.9.

Operations on Distributions

- Distributions are a vector space over $\mathbb{R}$:
  $(aF + bG)(\phi) = aF(\phi) + bG(\phi)$.
- Differentiation: $\partial_i F$ is the distribution defined by $\phi \mapsto -F(\partial_i \phi)$.

- Multiplication with a smooth function g: $(gF)(\phi) = F(g\phi)$.
- Convolution with test function g: $(g * F)(\phi) = F(\phi * \mathrm{P}g)$ where $\mathrm{P}g(x) = g(-x)$.
- $\delta * F = F$.
- Lemma 2.7: The convolution of a distribution corresponds to a smooth function.
- Lemma 2.8: for $f \in C(\Omega)$ we can undo the correspondence with $F_f(\lambda_{x,\epsilon}) \to f(x)$ as $\epsilon \to 0$.

Weak solutions

- If a function solves a PDE, its distribution also solves the PDE (in the sense of distributions).
- Are there other solutions if we look among distributions? This is the most general setting for the PDE.
- Allows you to consider discontinuous boundary conditions.
- You might find that the only distributions that solve the PDE correspond to functions.

Weak solutions to Transport and 1D wave - Ex18, Ex2.10, Ex42

- We have seen that solutions are $F(x-bt)$ and $F(x-t)+G(x+t)$ respectively when F and G are sufficiently differentiable.
- For all L^1_{loc} function the corresponding distributions are solutions.

Weak solutions to first order systems - S1.4, Ex5-7

- Section 1.4 we look for solutions to scalar conversation PDEs (Section 1.3): $\partial_t u + f'(u)\,\partial_x u = 0$ for $f : \mathbb{R} \to \mathbb{R}$. Particularly Burger's equation $f(u) = \frac{1}{2}u^2$.
- These PDEs are not linear, so distribution methods don't apply nicely.
- By method of characteristics, for some initial conditions no C^1 solution possible.
- Instead we look for solutions that are $C^1(\mathbb{R}^2)$ except for certain curves in the domain. We require that desirable Properties hold 'under the integral sign'.
- Theorem 1.11: $f \in C^2$ strictly convex, initial condition is bounded and L^1, then there is a unique solution of the scalar conservation PDE obeying Rakine-Hugonoit and Lax entropy conditions.