Chapter 4

Heat Equation

In this chapter we investigate the heat equation

$$\dot{u} - \triangle u = 0$$

and the corresponding inhomogeneous variant

$$\dot{u} - \triangle u = f.$$

The unknown function u is defined on an open domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and the inhomogeneity f is a given function on Ω. We shall extend some statements about harmonic functions to solutions of the heat equation.

This heat equation describes a diffusion process. This means a time-like evolution of space-like distributed quantities like heat, chemical concentration and others. Here the flow density is proportional to the negative of the gradient. Then the heat equation follows from the scalar conservation law.

4.1 Fundamental Solution

Since the heat equation is linear and contains only a first order derivative with respect to time and only second derivatives with respect to space, for any solution $u(x,t)$ and any $\lambda \in \mathbb{R}$ the function $u(\lambda x, \lambda^2 t)$ is also a solution. This scaling behaviour suggests to look for solutions which depend only on $\frac{x^2}{t}$. We invoke the following ansatz:

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad x \in \mathbb{R}^n, t \in \mathbb{R}^+.$$

Here α and β are constants and $v : \mathbb{R}^n \to \mathbb{R}$ an unknown function. This ansatz is justified by the scaling behaviour $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$. With $\lambda = \frac{1}{t}$ we obtain

$v(\frac{x}{t^\beta}) = u(\frac{x}{t^\beta}, 1)$. This ansatz transforms the heat equation into the following PDE

$$-\alpha \cdot t^{-(\alpha+1)}v(y) - \beta t^{-(\alpha+1)}y \cdot \nabla v(y) - t^{-(\alpha+2\beta)}\triangle v(y) = 0 \quad \text{mit} \quad y = \frac{x}{t^\beta}.$$

If we set $\beta = \frac{1}{2}$, then this equation does not depend on t and reduces to

$$\alpha v + \tfrac{1}{2}y \cdot \nabla v + \triangle v = 0.$$

Again we assume that v is a function of $|y|$. With $v(y) = w(|y|)$ we obtain:

$$\alpha w + \frac{1}{2}rw' + w'' + \frac{n-1}{r}w' = 0 \quad \text{with} \quad r = \frac{|\, x \,|}{\sqrt{t}}.$$

If we set $\alpha = \frac{n}{2}$, then we may integrate once:

$$\left(r^{n-1}w'\right)' + \tfrac{1}{2}\left(r^n w\right)' = 0 \qquad\qquad r^{n-1}w' + \tfrac{1}{2}r^n w = a.$$

The constant a vanishes, if w and w' vanish at infinity.

$$w' = -\tfrac{1}{2}rw \qquad\qquad w = b \cdot e^{-\frac{r^2}{4}}.$$

For a special choice of the constants a and b we again obtain the fundamental solution.

Definition 4.1. *The fundamental solution of the heat equation is defined as*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} & \textit{for} \quad x \in \mathbb{R}^n, t > 0 \\ 0 & \textit{for} \quad x \in \mathbb{R}^n, t < 0 \end{cases}.$$

Lemma 4.2. *For all $t > 0$ the fundamental solution satisfies* $\displaystyle\int_{\mathbb{R}^n} \Phi(x,t)d^n x = 1.$

Proof. $$\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}d^n x = \frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{-x^2}d^n x = \frac{1}{\pi^{n/2}}\left(\int_{\mathbb{R}} e^{-x^2}dx\right)^n = 1.$$ **q.e.d.**

The fundamental solution is similar to a mollifier on $\mathbb{R}^n$. So we may expect that the convolution with Φ converges in the limit $t \downarrow 0$ like the identity.

Theorem 4.3. *For $h \in C_b(\mathbb{R}^n, \mathbb{R})$ the following function u has the properties (i)-(iii):*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)h(y)d^n y$$

(i) $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$

(ii) $\dot{u} - \triangle u = 0$ *on* $\mathbb{R}^n \times \mathbb{R}^+$

(iii) u *extends continuously and bounded to* $\mathbb{R}^n \times [0, \infty)$ *with* $\lim_{t \to 0} u(x,t) = h(x)$.

Proof. Since $\Phi(x,t)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^+$ the foregoing lemmas and the boundedness of h implies that $u(x,t)$ is well defined, bounded and continuous on $\mathbb{R}^n \times [0,\infty)$. On $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ all partial derivatives of $(x,t) \mapsto \Phi(x-y,t)$ belong to $L^1(\mathbb{R}^n)$ considered as functions on $y \in \mathbb{R}^n$ and depend continuously on $(x,t) \in \mathbb{R}^n$. So they define a smooth map from $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ into $L^1(\mathbb{R}^n)$. The integral is a linear continuous operator from $L^1(\mathbb{R}^n)$ to $\mathbb{R}$. So u is smooth. No (ii) follows, since Φ solves the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$. The continuity of h implies uniform continuity on compact subsets. For any $\epsilon > 0$ and any x in a compact subset of $\mathbb{R}^n$ there exists $\delta > 0$, such that $|h(x) - h(y)| < \epsilon$ for all $|x - y| < \delta$. Furthermore there exists $T > 0$, such that

$$\int_{\mathbb{R}^n \setminus B(0,\delta)} \Phi(y,t) d^n y = \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} \Phi(y,1) d^n y < \epsilon \qquad \text{for all } t < T.$$

This implies $\qquad |u(x,t) - h(x)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(h(y) - h(x)) d^n y \right|$

$$\leq \int_{B(x,\delta)} \Phi(x-y,t) \mid h(y) - h(x) \mid d^n y \quad + \int_{\mathbb{R}^n \setminus B(x,\delta)} \Phi(x-y,t)|h(y) - h(x)| d^n y$$
$$\leq \epsilon + 2\epsilon \sup\{|h(y)| \mid y \in \mathbb{R}^n\} \qquad \text{for all } t < T.$$

So $u(x,t)$ converges in the limit $t \downarrow 0$ uniformly on compact subsets of $\mathbb{R}^n$ to h. **q.e.d.**

In this limit $t \downarrow 0$ Φ converges as a distribution (and as a measure) to the δ-distribution. Note that by this formula the speed of propagation is unbounded.

4.2 Inhomogeneous Initial value problem

In the forgoing section we constructed a solution of the initial value problem

$$\dot{u} - \triangle u = 0 \quad \text{and} \quad u(x,0) = h(x).$$

Duhamel's principle derives solutions of the inhomogeneous initial value problem from solutions of the homogeneous initial values problem. If we write the heat equation as $\dot{u} = \triangle u$ and recall that the Laplace operator is a linear map from the space of smooth functions on $\mathbb{R}^n$ into itself, then the heat equation becomes a linear ODE in the (infinite-dimensional) space of smooth functions on $\mathbb{R}^n$. For linear ODEs the variation of constants is also a method to obtain the solutions of the inhomogeneous equation in terms of homogeneous solutions. In fact if we take the integral over the interval $[0,t]$ of the corresponding homogeneous solutions which are at $s \in [0,t]$ equal to the inhomogeneity at s, then we obtain a solution of the inhomogeneous equation which vanishes at $t=0$. Now Duhamel's principle is just the application of the variation of constants to the heat equation considered as an ODE in the space of functions on $\mathbb{R}^n$:

Let $\quad u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds.\quad$ Then formally we obtain

$$\begin{aligned}\dot{u}(x,t) - \triangle u(x,t) &= \lim_{s\to 0}\int_{\mathbb{R}^n} \Phi(x-y,s)f(y,t-s)d^n y + \\ &+ \int_0^t \int_{\mathbb{R}^n} \Big(\dot{\Phi}(x-y,t-s) - \triangle_x \Phi(x-y,t-s)\Big) f(y,s)d^n y ds = f(x,t).\end{aligned}$$

Theorem 4.4 (Solution of the inhomogeneous initial value problem)**.** *If f is twice continuously and bounded differentiable on $\mathbb{R}^n \times [0,\infty)$, then*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f(x-y,t-s)d^n y ds$$

solves the inhomogeneous initial value problem

$$\dot{u} - \triangle u = f \text{ on } \mathbb{R}^n \times \mathbb{R}^+ \text{ and } \qquad \lim_{t\to 0} u(x,t) = 0.$$

Proof. We already proved that $v_s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y$ solves on $\mathbb{R}^n \times (s,\infty)$ the initial value problem $\dot{v_s} - \triangle v_s = 0$ with $\lim_{t\to s} v_s(x,t) = f(x,t)$. So v_s is on $\mathbb{R}^n \times [s,\infty)$ continuous. This implies for all $\epsilon > 0$ the relation

$$u_\epsilon(x,t) = \int_0^{t-\epsilon} v_s(x,t)ds = \int_0^{t-\epsilon} \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds$$

$$\dot{u}_\epsilon(x,t) - \triangle u_\epsilon(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t-(t-\epsilon))f(y,t-\epsilon)d^n y = \int_{\mathbb{R}^n} \Phi(x-y,\epsilon)f(y,t-\epsilon)d^n y.$$

Theorem 4.3 (iii) implies $\lim_{\epsilon\to 0} \dot{u}_\epsilon - \triangle u_\epsilon = f$ on $\mathbb{R}^n \times \mathbb{R}^+$. On the other hand we have

$$u_\epsilon(x,t) = \int_0^{t-\epsilon} \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds = \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)f(x-y,t-s)d^n y ds.$$

By the second integral in the Theorem and the assumptions on f we conclude that

$$\lim_{\epsilon\to 0}\left(\dot{u}_\epsilon(x,t) - \triangle u_\epsilon(x,t)\right) = \left(\frac{\partial}{\partial t} - \triangle\right)\lim_{\epsilon\to 0} u_\epsilon(x,t) = \left(\frac{\partial}{\partial t} - \triangle\right)u(x,t)$$

holds. The continuity of v gives $u(x,0) = 0$. **q.e.d.**

Corollary 4.5. *The inhomogeneous initial value problem has the following solution:*

$$\dot{u} - \triangle u = f \qquad\qquad u(x,0) = h(x)$$

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)h(y)d^n y \;\; + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds. \qquad \textbf{q.e.d.}$$

4.3 Mean Value Property

We use the fundamental solution $\Phi(x,t)$ in order to determine the value $u(x,t)$ as a mean value on some ball like domain which has to be chosen properly.

Definition 4.6. *For all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and all $r > 0$ we define*

$$E(x,t,r) = \left\{(y,s) \in \mathbb{R}^{n+1} \mid s \leq t, \Phi(x-y,t-s) \geq \tfrac{1}{r^n}\right\}$$

$$\begin{aligned} e^{-\frac{|x-y|^2}{4(t-s)}} \geq \frac{(4\pi)^{n/2}(t-s)^{n/2}}{r^n} \quad &\iff \quad e^{\frac{|x-y|^2}{4(t-s)}} \leq \frac{1}{\pi^{n/2}} \left(\frac{r^2}{4(t-s)}\right)^{n/2} \\ &\iff \quad \frac{\mid x-y \mid^2}{4(t-s)} \leq \frac{n}{2}(2\ln(r) - \ln(4(t-s)) - \ln(\pi)) \\ &\iff \quad \mid x-y \mid^2 \leq 2(t-s)n(2\ln(r) - \ln(t-s) - \ln(4\pi)). \end{aligned}$$

Theorem 4.7 (mean value property of the heat equation)**.** *Let u be a solution of the heat equation on an open domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$. For any $(x,t) \in \Omega$ and any $r > 0$ with $E(x,t,r) \subset \Omega$ we have*

$$u(x,t) = \frac{1}{C_n r^n} \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d^n y ds \qquad \text{with} \qquad C_n = \int_{E(0,0,1)} \frac{|y|^2}{s^2} d^n y ds.$$

Proof. Due to the translation invariance we may assume $(x,t) = (0,0)$. We define

$$\phi(r) = \frac{1}{r^n} \int\limits_{E(0,0,r)} u(z,q) \frac{|z|^2}{q^2} d^n z dq = \frac{1}{r^n} \int\limits_{E(0,0,r)} u(ry, r^2 s) \frac{|ry|^2}{(r^2 s)^2} d^n(ry) d(r^2 s) = \int\limits_{E(0,0,1)} u(ry, r^2 s) \frac{|y|^2}{s^2} d^n y ds.$$

Here we used the fact that the bijective map $(y,s) \mapsto (ry, r^2 s)$ maps $E(x,t,1)$ onto $E(rx, r^2 t, r)$ since $\Phi(r(x-z), r^2 t) = r^{-n}\Phi(x-y,t)$. We calculate

$$\begin{aligned} \phi'(r) &= \int_{E(0,0,1)} \frac{|y|^2}{s^2} \left(y \cdot \nabla u(ry, r^2 s) + 2rs\dot{u}(ry, r^2 s)\right) d^n y ds \\ &= \frac{1}{r^{n+1}} \int_{E(0,0,r)} \frac{|y|^2}{s^2} y \cdot \nabla u(y,s) d^n y ds + \frac{1}{r^{n+1}} \int_{E(0,0,r)} 2\dot{u}(y,s) \frac{\mid y \mid^2}{s} d^n y ds \end{aligned}$$

For $\psi = -\frac{n}{2}\ln(-4\pi s) + \frac{|y|^2}{4s} + n\ln r$ we obtain $E(0,0,r) = \{(y,s) \mid \psi(y,s) \geq 0\}$. Furthermore ψ vanishes on the boundary of $E(0,0,r)$.

$$\begin{aligned}
\frac{1}{r^{n+1}}\int_{E(0,0,r)} 2\dot{u}\frac{|\, y\,|^2}{s}d^nyds &= \frac{1}{r^{n+1}}\int_{E(0,0,r)} 4\dot{u}y\cdot\nabla\psi d^nyds \\
&= -\frac{1}{r^{n+1}}\int_{E(0,0,r)} (4n\dot{u}\psi + 4\psi y\cdot\nabla\dot{u})d^nyds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} (-4n\dot{u}\psi + 4\dot{\psi}y\cdot\nabla u)d^nyds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \left(-4n\dot{u}\psi + 4\left(-\frac{n}{2s} - \frac{|\, y\,|^2}{4s^2}\right)y\cdot\nabla u\right)d^nyds.
\end{aligned}$$

Hence we have
$$\begin{aligned}
\phi'(r) &= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \left(-4n\triangle u\psi - \frac{2n}{s}y\cdot\nabla u\right)d^nyds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \left(4n\nabla u\cdot\nabla\psi - \frac{2n}{s}y\cdot\nabla u\right)d^nyds = 0.
\end{aligned}$$

This shows that ϕ is constant. By the continuity of u and by the equation

$$\frac{1}{r^n}\int_{E(0,0,r)} \frac{|\, y\,|^2}{s^2}d^nyds = \frac{1}{r^n}\int_{E(0,0,r)} \frac{|\, ry\,|^2}{(r^2s)^2}d^nrydr^2s = \int_{E(0,0,1)} \frac{|\, y\,|^2}{s^2}d^nyds = C_n$$

we obtain $\lim_{r\to 0}\phi(r) = C_nu(0,0)$. **q.e.d.**

It is possible to calculate the constant explicitly. The heat ball $E(0,0,1)$ contains all $(y,s) \in \mathbb{R}^n \times (-\infty, 0]$ with $s \leq 0$ and $|y|^2 \leq -2sn(2\ln(1) - \ln(-s) - \ln(4\pi)) = 2ns\ln(-4\pi s)$. By the positivity of $|y|^2$ we have $-4\pi s < 1$ and $-\frac{1}{4\pi} < s < 0$. This gives

$$\begin{aligned}
C_n &= \int_{-\frac{1}{4\pi}}^{0}\frac{1}{s^2}\int_{B(0,\sqrt{2ns\ln(-4\pi s)}\subset\mathbb{R}^n} |y|^2d^nyds \\
&= \int_{-\frac{1}{4\pi}}^{0}\frac{1}{s^2}\int_0^{\sqrt{2ns\ln(-4\pi s)}} n\omega_n r^{n+1}drds = \int_0^{\frac{1}{4\pi}}\frac{1}{s^2}\int_0^{\sqrt{-2ns\ln(4\pi s)}} n\omega_n r^{n+1}drds \\
&= n\omega_n\int_0^{\frac{1}{4\pi}}\frac{1}{s^2}\left[\frac{r^{n+2}}{n+2}\right]_0^{\sqrt{2ns\ln\left(\frac{1}{4\pi s}\right)}} ds = \frac{n\omega_n(2n)^{\frac{n+2}{2}}}{n+2}\int_0^{\frac{1}{4\pi}}\left(s\ln\left(\tfrac{1}{4\pi s}\right)\right)^{\frac{n+2}{2}}\frac{ds}{s^2}.
\end{aligned}$$

Now we substitute $4\pi s = e^{-\frac{2}{n}t^2}$ with $\frac{2}{n}t^2 = \ln(\frac{1}{4\pi s})$ and $\frac{4}{n}tdt = -\frac{ds}{s}$.

$$\begin{aligned}
C_n &= \frac{n\omega_n(2n)^{\frac{n+2}{2}}}{n+2}\int_0^\infty \left(\frac{e^{-\frac{2}{n}t^2}}{4\pi}\right)^{\frac{n}{2}} \left(\tfrac{2}{n}t^2\right)^{\frac{n+2}{2}} \tfrac{4}{n}tdt = \frac{n\omega_n 2^{n+2-n+1}}{(n+2)n\pi^{\frac{n}{2}}}\int_0^\infty 2te^{-t^2}t^{n+2}dt\\
&= -\frac{8\omega_n}{(n+2)\pi^{\frac{n}{2}}}\left[e^{-t^2}t^{n+2}\right]_0^\infty + \frac{8\omega_n}{\pi^{\frac{n}{2}}}\int_0^\infty e^{-t^2}t^{n+1}dt = \frac{4\omega_n}{\pi^{\frac{n}{2}}}\int_0^\infty 2te^{-t^2}t^n dt\\
&= -\frac{4\omega_n}{\pi^{\frac{n}{2}}}\left[e^{-t^2}t^n\right]_0^\infty + \frac{4}{\pi^{\frac{n}{2}}}\int_0^\infty n\omega_n e^{-t^2}t^{n-1}dt = \frac{4}{\pi^{\frac{n}{2}}}\left(\int_{-\infty}^\infty e^{-x^2}dx\right)^n = 4.
\end{aligned}$$

4.4 Maximum Principle

For any open domain $\Omega \subset \mathbb{R}^n$ we define the parabolic cylinder as $\Omega_T = \Omega \times (0, T]$. The parabolic boundary $\partial\Omega_T$ of Ω_T is defined as $\bar{\Omega}_T \setminus \Omega_T$. It is the union of $(\partial\Omega \times (0, T]) \cup \big(\bar{\Omega} \times 0\big)$ and does not contain at time $t = T$ points inside of Ω.

Theorem 4.8 (strong maximum principle of the heat equation)**.** *Let Ω be path connected (i.e. any $x, x' \in \Omega$ are connected by a continuous path from x to x') and let u be twice continuously differentiable solution of the heat equation on Ω_T with continuous extension to $\bar{\Omega}_T$. If u takes the maximal value in Ω_T, then u is constant on $\bar{\Omega}_T$.*

Proof. Let (x_0, t_0) be an element of Ω_T at which u takes the maximal value. Then there exists $r_0 > 0$ such that $E(x_0, t_0, r_0)$ is contained in Ω_T. By the mean value property u is constant on $E(x_0, t_0, r_0)$. Since Ω is path connected there exists for any $(x, t) \in \Omega \times (0, t_0)$ finitely many $E(x_0, t_0, r_0), E(x_1, t_1, r_1), \ldots, E(x_n, t_n, r_n)$ in $\Omega \times (0, t_0)$ containing the points $(x_1, t_1), \ldots, (x_n, t_n), (x, t)$. So u is constant on $\bar{\Omega}_T$. **q.e.d.**

Theorem 4.9 (weak maximum prinziple for the heat equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and u a twice differentiable solution of the heat equation on Ω_T which extends continuously to $\bar{\Omega}_T$. Then the maximum of u is taken on $\partial\Omega_T$.* **q.e.d.**

Again this Maximum principle implies the uniqueness of a boundary value problem:

Theorem 4.10 (uniqueness of the boundary value problem)**.** *On an open and bounded domain $\Omega \subset \mathbb{R}^n$ there exists at most one solution u of the inhomogeneous heat equation which extends continuously to $\bar{\Omega}_T$ and coincides on $\partial\Omega_T$ with a given function.*

Proof. Apply the weak maximum principle to the difference of two solutions. **q.e.d.**

In order to prove on $\mathbb{R}^n \times \mathbb{R}^+$ the uniqueness of the initial value problem we need as in the case of the Poisson problem a bound on the growth at infinity.

Theorem 4.11 (maximum prinziple for the Cauchy problem)**.** *For a bounded and continuous initial value h on $\mathbb{R}^n$ let u be a solution on $\mathbb{R}^n \times (0,T]$ of the problem:*

$$\dot{u} - \triangle u = 0 \text{ on } \mathbb{R}^n \times (0,T) \qquad\qquad u(x,0) = h(x) \text{ on } \mathbb{R}^n \times \{0\},$$

which is bounded by $\qquad u(x,t) \leq Ae^{a|x|^2} \quad \text{on} \quad \mathbb{R}^n \times [0,T]$
for some positive constants $A, a > 0$. Then u is bounded by $\qquad \sup\limits_{\mathbb{R}^n \times [0,T]} u = \sup\limits_{\mathbb{R}^n} h.$

Proof. We first consider the case where a and T obey $4aT < 1$. Then there exists an $\epsilon > 0$ with $4a(T + \epsilon) < 1$. For all $y \in \mathbb{R}^n$ and $\mu > 0$ the following function v solves together with the fundamental solution on $\mathbb{R}^n \times (0, T + \epsilon)$ the heat equation:

$$v(x,t) = u(x,t) - \mu(T + \epsilon - t)^{-\frac{n}{2}} \exp\left(\tfrac{|x-y|^2}{4(T+\epsilon-t)}\right)$$

On any domain of the form $\Omega_T = B(y,r) \times (0,T]$ the weak maximum principle applies. Due to the assumptions both function u and h are bounded by $Ae^{a|x|^2}$. Since the inequality $\frac{1}{4(T+\epsilon-t)} > a$ holds for $t > 0$ there exists for any $\mu > 0$ a $R > 0$ such that $v(x,t) \leq \sup\{h(x) \mid x \in \mathbb{R}\}$ holds for all $r > R$ on $\partial B(y,r)_T = B(y,r) \times \{0\} \cup \partial B(y,r) \times (0,T]$. Hence the weak maximum principle implies $v(x,t) \leq \sup\{h(x) \mid x \in \mathbb{R}^n\}$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. This holds for all $\mu > 0$ and by continuity also for $\mu = 0$.

For $4aT \geq 1$ we decompose the time interval into $[0,T] = [0,T_1] \cup \ldots \cup [T_M, T]$ with the property $4a(T_{m+1} - T_m) < 1$. By induction the general case follows. **q.e.d.**

Theorem 4.12 (existence and uniqueness of the initial value problem)**.** *For $h \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$ there exists at most one solution of the initial value problem*

$$\dot{u} - \triangle u = f \text{ on } \mathbb{R}^n \times (0,T) \qquad\qquad u = h \text{ on } \mathbb{R}^n \times \{0\}$$

on $\mathbb{R}^n \times [0,T_0]$ which is bounded by $|u(x,t)| \leq Ae^{a|x|^2}$ for some $A > 0$, $a > 0$ and $0 < T_0 \leq T$.

If h and f are bounded by $|h(x)| \leq Ae^{a|x|^2}$ and $f(x,t) \leq Ae^{a|x|^2}$ on $(x,t) \in \mathbb{R}^n \times [0,T]$ for some $A > 0$, $a > 0$, and $T > 0$ then this unique solution is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) h(y) d^n y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) d^n y ds.$$

This solution might explode at some finite $t \uparrow T_0 \geq \frac{1}{16a}$.

Proof. By the maximum principle for for the Cauchy problem Theorem 4.11 the difference of any two solutions vanishes. This shows uniqueness.

In order to prove existence we apply Corollary 4.5 and show that the given $u(x,t)$ has a bound as stated. For $0 \le t-s \le \frac{1}{16a}$ we have $-\frac{|x-y|^2}{4(t-s)} \le -2a|x-y|^2 - \frac{|x-y|^2}{8(t-s)}$ and

$$e^{-2a|x|^2}e^{a|y|^2}\Phi(x-y,t-s) \le \frac{e^{-2a|x|^2+a|y|^2-2a|x-y|^2}e^{-\frac{|x-y|^2}{8(t-s)}}}{(4\pi(t-s))^{n/2}} = \frac{2^{n/2}e^{-a|2x-y|^2}e^{-\frac{|x-y|^2}{8(t-s)}}}{(8\pi(t-s))^{n/2}},$$
$$\Phi(x-y,t-s) \le 2^{n/2}\Phi(x-y,2(t-s))e^{2a|x|^2}e^{-a|y|^2}.$$

The inequalities $|h(x)| \le Ae^{a|x|^2}$ and $f(x,t) \le Ae^{a|x|^2}$ which hold for $(x,t) \in \mathbb{R}^n \times [0,T]$ first imply $u(x,t) \le A'e^{2a|x|^2}$ for $t \in [0,T_0]$ with $T_0 = \min\{T, \frac{1}{16a}\}$ and some $A' > 0$. For $f = 0$ the maximum principle for the Cauchy problem Theorem 4.11 implies

$$\sup_{(x,t)\in\mathbb{R}^n\times[0,T_0]} e^{-2a|x|^2}|u(x,t)| \le 2^{\frac{n}{2}} \sup_{(x,t)\in\mathbb{R}^n\times[0,T_0]} \int_{\mathbb{R}^n}\Phi(x-y,2t)e^{-a|y|^2}|h(y)|d^ny$$
$$\le 2^{\frac{n}{2}} \sup_{y\in\mathbb{R}^n} e^{-a|y|^2}|h(y)| \le 2^{\frac{n}{2}}A.$$

For non vanishing f we get $\sup\limits_{(x,t)\in\mathbb{R}^n\times[0,T_0]} e^{-2a|x|^2}|u(x,t)| \le 2^{\frac{n}{2}}A\left(1+\int\limits_0^t ds\right) \le 2^{\frac{n}{2}}A(1+T)$. So the given u obeys locally in $t \in [0,T]$ a bound as stated and is the unique solution, as long as it obeys such a bound. The solution $u(x,t) = (T_0-t)^{-\frac{n}{2}}\exp\left(\frac{|x|^2}{4(T_0-t)}\right)$ of the homogeneous heat equation shows that this might not be true for all $t \in [0,T]$. **q.e.d.**

Improved arguments yields the sharp bound on the extinction time $T_0 \ge \frac{1}{4a}$.

Example 4.13. *We show by a counterexample the non uniqueness of solutions without any bound of the initial value problem. For $n = 1$ we make the ansatz*

$$u(x,t) = \sum_{l=0}^{\infty} g_l(t)x^l, \qquad \dot{u}(x,t) - \triangle u(x,t) = \sum_{l=0}^{\infty}(\dot{g}_l(t) - (l+2)(l+1)g_{l+2}(t))x^l.$$

For a given function $g_0(t) = g(t)$ we thus obtain the following formal solution of the homogeneous heat equation:

$$u(x,t) = \sum_{l=0}^{\infty} \frac{g^{(l)}(t)}{(2l)!}x^{2l}.$$

We now show that for $g(t) = \exp(-t^{-2})$ this power series indeed converges to a solution such that on every compact subset of $\mathbb{R}^n$ the uniform limit $t \downarrow 0$ vanishes. We first calculate $g^{(l)}(t)$ for any $l \in \mathbb{N}_0$ by a real polynomial p_l of degree l solving the relation

$$g^{(l)}(t) = t^{-l}p_l(t^{-2})\exp(-t^{-2}) \quad \textit{with} \quad p_{l+1}(z) = 2zp_l(z) - lp_l(z) - 2zp_l'(z).$$

This recursion relation for p_l follows by differentiating by t. The first two polynomials are $p_0(z) = 1$ and $p_1(z) = 2z$. We claim that the coefficient of $p_l(z)$ in front of z^k is bounded by $\frac{l!7^l}{2^k k!}$. For $l = 0$, $k = 0$ this is clear. By induction we obtain with $k \leq l+1$

$$2\frac{l!7^l}{2^{k-1}(k-1)!} + l\frac{l!7^l}{2^k k!} + 2k\frac{l!7^l}{2^k k!} = \frac{l!7^l(4k+l+2k)}{2^k k!} \leq \frac{l!7^l 7(l+1)}{2^k k!} \leq \frac{(l+1)!7^{l+1}}{2^k k!}.$$

This proves the claim. Using the inequalities $\frac{l!}{(2l)!} = \frac{1}{2^l 1\cdot 3 \cdots (2l-1)} \leq \frac{1}{2^l l!}$ we conclude

$$|u(x,t)| \leq \sum_{l=0}^{\infty} \frac{l!7^l x^{2l}}{(2l)!t^l} \sum_{k=0}^{l} \frac{g(t)}{2^k k! t^{2k}} \leq \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{7x^2}{2t}\right)^l \sum_{k=0}^{\infty} \frac{g(t)}{k!}\left(\frac{1}{2t^2}\right)^k = \exp\left(\frac{7x^2}{2t} - \frac{1}{2t^2}\right).$$

Therefore the series converges absolutely and for $t \downarrow 0$ uniformly on compact sets to 0.

In analogy to the Laplace equation one can show the uniqueness of the boundary value problem Theorem 4.10 and of the initial value problem Theorem 4.12 also with the monotonicity of an energy functional. We define

$$e(t) = \int_\Omega u^2(x,t) d^n x.$$

If u solves the homogeneous heat equation and vanishes at the boundary of Ω, then this functional is monotonically decreasing with respect to time:

$$\dot{e}(t) = 2\int_\Omega u(x,t)\dot{u}(x,t)d^n x = 2\int_\Omega u(x,t)\triangle u(x,t) d^n x = -2\int_\Omega (\nabla u(x,t))^2 d^n x \leq 0.$$

If $u(x,t)$ vanishes at $t = 0$, and if $u(\cdot,t)$ and $\nabla u(\cdot,t)$ are square integrable for $t > 0$, then u vanishes identically since $\nabla u(\cdot,t)$ vanishes and $u(\cdot,t)$ is constant for $t > 0$.

4.5 Heat Kernel

In analogy to the Green's function of the Laplace equation we define for open subsets $\Omega \subset \mathbb{R}^n$ the heat kernel H_Ω.

Definition 4.14. *For an open domain $\Omega \subset \mathbb{R}^n$ the heat kernel $H_\Omega : \Omega \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$ of Ω is characterised by the following two properties:*

(i) *For $(x,t) \in \Omega \times \mathbb{R}^+$ $y \mapsto H_\Omega(x,y,t)$ extends continuously to $\bar{\Omega}$ with value 0 on $\partial\Omega$.*

(ii) *For $x \in \Omega$ the function $(y,t) \mapsto H_\Omega(x,y,t) - \Phi(x-y,t)$ solves the homogeneous heat equation and extends continuously to $\bar{\Omega} \times \mathbb{R}^+_0$ with value 0 on $(y,t) \in \bar{\Omega} \times \{0\}$.*

Lemma 4.15. *If u and v are two functions on $\Omega \times \mathbb{R}^+$ with an open domain $\Omega \subset \mathbb{R}^n$ which all three have appropriate regularity, then we have*

$$\begin{aligned}
& \int_0^T \int_\Omega u(x,t)(\partial_t v(x,T-t) + \triangle v(x,T-t))d^n x dt \\
+ & \int_0^T \int_\Omega (\partial_t u(x,t) - \triangle u(x,t)) v(x,T-t) d^n x dt = \\
= & \int_0^T \int_{\partial\Omega} (u(y,t)\nabla_y v(y,T-t) - \nabla_y u(y,t) v(y,T-t)) \cdot N(y) d\sigma(y) dt \\
+ & \int_\Omega (u(x,T)v(x,0) - u(x,0)v(x,T)) d^n x.
\end{aligned}$$

Proof. The fundamental theorem of calculus gives for the terms with t-derivatives the final integral over Ω and the boundary terms of a partial integration with respect to y yields the two gradients with respect to x in the integral over $\partial\Omega$. **q.e.d.**

The function $v(y,t) = H_\Omega(x,y,t)$ has at $v(x,0)$ a singularity and is not defined there. Hence we integrate with respect to dt over the interval $t \in [0, T-\epsilon]$ instead of $t \in [0,T]$ and take afterwards the limit $\epsilon \downarrow 0$. Then Theorem 4.3 gives

$$\begin{aligned}
& \int_0^T \int_\Omega (\dot{u}(y,t) - \triangle u(y,t)) H_\Omega(x,y,T-t) d^n y dt = \\
= & \int_0^T \int_{\partial\Omega} u(z,t) \nabla_z H_\Omega(x,z,T-t) \cdot N(z) d\sigma(z) dt + u(x,T) - \int_\Omega u(y,0) H_\Omega(x,y,T) d^n y.
\end{aligned}$$

This shows also

$$\begin{aligned}
u(x,T) = & \int_0^T \int_\Omega (\dot{u}(y,t) - \triangle u(y,t)) H_\Omega(x,y,T-t) d^n y dt \\
& - \int_0^T \int_{\partial\Omega} u(z,t) \nabla_z H_\Omega(x,z,T-t) N(z) d\sigma(z) dt + \int_\Omega u(y,0) H_\Omega(x,y,T) d^n y.
\end{aligned}$$

Theorem 4.16 (solution of the initial and boundary value problem)**.** *Let f be a function on $\Omega \times (0,T)$, g a function on $\partial\Omega \times [0,T]$ and h a function on Ω which together with the open domain $\Omega \subset \mathbb{R}^n$ have appropriate regularity such that all appearing integrals converge absolutely. Then*

$$\begin{aligned}
u(x,T) = & \int_0^T \int_\Omega f(y,t) H_\Omega(x,y,T-t) d^n y dt \\
& - \int_0^T \int_{\partial\Omega} g(z,t) \nabla_z H_\Omega(x,z,T-t) N(z) d\sigma(z) dt + \int_\Omega h(y) H_\Omega(x,y,T) d^n y
\end{aligned}$$

is the unique solution of the initial and boundary value problem

$$\dot{u} - \triangle u = f \text{ on } \Omega \times (0,T) \qquad u = g \text{ on } \partial\Omega \times [0,T] \qquad u(x,0) = h(x) \text{ on } \Omega.$$

We prepare the proof by showing that the heat kernel is symmetric:

Lemma 4.17. *For all $T > 0$ and $x, y \in \bar{\Omega}$ we have $H_\Omega(x,y,T) = H_\Omega(y,x,T)$.*

Proof. We insert $u(z,t) = H_\Omega(x,z,t)$ and $v(z,t) = H_\Omega(y,z,t)$ in Lemma 4.15. By Theorem 4.3 (iii) and the property (ii) of the heat kernel the following integral vanishes:

$$\int_\Omega (H_\Omega(x,z,T)H_\Omega(y,z,0) - H_\Omega(x,z,0)H_\Omega(y,z,T))d^nz = H_\Omega(x,y,T) - H_\Omega(y,x,T).\textbf{q.e.d.}$$

Sketch of the proof of Theorem 4.16. The case $f = 0 = g$ follows from the defining properties of the heat kernel. This implies that in the second case $g = 0 = h$ $v(x,T) = \int_\Omega H_\Omega(x,y,T-t)f(y,t)d^ny$ solves the initial value problem

$$\dot{v} - \triangle v = 0 \text{ on } \Omega \times (t,\infty) \quad v(x,t) = f(x,t) \text{ on } \Omega \times \{t\} \quad v(x,t) = 0 \text{ on } \partial\Omega \times [0,\infty].$$

If we assume that f has appropriate regularity and extends twice continuously differentiable to $\bar{\Omega} \times [0,T]$ as in the homogeneous initial value problem Theorem 4.4, then

$$u(x,T) = \int_0^T \int_\Omega H_\Omega(x,y,T-t)f(y,t)d^nydt \qquad \text{solves the initial value problem}$$

$$\dot{u}(x,t) - \triangle u(x,t) = f \text{ on } \Omega \times (0,T) \quad u(x,0) = 0 \text{ on } \Omega \quad u(x,t) = 0 \text{ on } \partial\Omega \times [0,T].$$

Finally we consider the inhomogeneous boundary value problem: In this case u solves

$$\dot{u}(x,t) - \triangle u(x,t) = 0 \text{ on } \Omega \times (0,T) \quad u(x,0) = 0 \text{ on } \Omega \quad u(x,t) = g \text{ on } \partial\Omega \times [0,T].$$

We first extend any function g on $\partial\Omega \times [0,T]$ with appropriate regularity to $\Omega \times [0,T]$ such that it vanishes outside a tubular neighbourhood of $\partial\Omega \times [0,T]$. If we subtract from this extension $\tilde{u}$ the solution of $f = \dot{\tilde{u}} - \triangle\tilde{u}$ and $h(x) = \tilde{u}(x,0)$ then we obtain a solution of the desired boundary value problem. **q.e.d.**

The appropriate regularity conditions depend on the heat kernel and therefore also on the domain. All the time we assumed that the divergence theorem holds for the open domain $\Omega \subset \mathbb{R}^n$. Before we construct the heat kernel for some special domains, we prove the following general property of the heat kernel:

Lemma 4.18. *For any bounded connected open domain $\Omega \subset \mathbb{R}^n$ the corresponding heat kernel is positive on the corresponding parabolic cylinder, if it exists.*

Proof. The fundamental solution $\Phi(x,t)$ is positive on $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. For bounded open domains $\Omega \subset \mathbb{R}^n$ and given $x \in \Omega$ the difference $\Phi(x-y,t) - H_\Omega(x,y,t)$ of the fundamental solution minus the heat kernel is the unique solution of the heat equation on $\Omega \times [0,T]$ which vanishes on $\Omega \times \{t=0\}$ and coincides on $\partial\Omega \times [0,T]$ with $\Phi(x-y,t)$. This solution is for all $\epsilon > 0$ on $\Omega \times \{t=\epsilon\}$ and on $\partial\Omega \times [0,T]$ not larger than $\Phi(x-y,t)$. By the Maximum Principle it is not larger than $\Phi(x-y,t)$ and $H_\Omega(x,y,t)$ is positive. **q.e.d.**

4.6 Spectral Theory and the Heat Equation

In this section we solve the initial value problem

$$\dot{u} - \triangle u = 0 \quad \text{on} \quad \Omega \times [0,T] \qquad u = 0 \quad \text{on } \partial\Omega \times [0,T] \qquad u = h \quad \text{on } \Omega \times \{0\}$$

with the help of the Laplace operator on Ω. If h is an eigenfunction of the Laplace operator:

$$-\triangle h = \lambda h \quad \text{on} \quad \Omega \quad \text{and} \quad h|_{\partial\Omega} = 0,$$

then the initial value problem can be solved by the following ansatz:

$$u(x,t) = \varphi(t)h(x) \quad \dot{\varphi}(t)h(x) + \lambda\varphi(t)h(x) = 0.$$

The general solution is $\dot{\varphi} = -\lambda\varphi$, $\varphi(t) = e^{-\lambda(t-t_o)}$. With $\varphi(0) = 1$ we obtain the unique solution of the corresponding initial value problem $u(x,t) = e^{-\lambda t}h(x)$. By linearity the corresponding solution for initial value $h = h_1 + \ldots + h_M$ with $-\triangle h_i = \lambda_i h_i$ on Ω and $h_i|_{\partial\Omega} = 0$ is given by $u(x,t) = e^{-\lambda_1 t}h_1(x) + \ldots + e^{-\lambda_M t}h_M(x)$. Hence it suffices to decompose h into a sum of eigenfunctions of the Laplace operator on Ω with Dirichlet boundary conditions.

To explain this strategy we first interpret the fundamental solution as such a decomposition. On $\mathbb{R}^n$ the Laplace operator has the following eigenfunctions:

$$-\triangle e^{2\pi i k\cdot x} = 4\pi^2 k^2 e^{2\pi i k x}.$$

The following equation of powerseries holds first for $x \in \mathbb{R}^n$ and then also for $x \in i\mathbb{R}^n$:

$$\pi^{n/2}\sum_{l=0}^{\infty}\frac{(x^2)^l}{l!} = \int_{\mathbb{R}^n} e^{-(k-x)^2+x^2}d^nk = \int_{\mathbb{R}^n} e^{-k^2}\sum_{l=0}^{\infty}\frac{(2kx)^l}{l!}d^nk = \sum_{l=0}^{\infty}\frac{1}{(2l)!}\int_{\mathbb{R}^n} e^{-k^2}(2kx)^{2l}d^nk.$$

This implies $\displaystyle\frac{\pi^{n/2}}{(4\pi^2 t)^{n/2}}e^{-\frac{(x-y)^2}{4t}} = \int_{\mathbb{R}^n} e^{-\left(2\pi k\sqrt{t} - i\frac{x-y}{2\sqrt{t}}\right)^2} e^{-\frac{(x-y)^2}{4t}}d^nk = \int_{\mathbb{R}^n} e^{-4\pi^2k^2t}e^{2\pi i(x-y)k}d^nk.$

So by our considerations above the solution of the initial value problem

$$\dot{u} - \triangle u = 0 \quad \text{on} \quad \mathbb{R}^n \times [0, T] \qquad\qquad u(x, 0) = h \quad \text{on} \quad \mathbb{R}^n$$

is given by

$$u(x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i (x-y)k} h(y) d^n k d^n y.$$

For an integrable function h we can apply Fubini's Theorem. So for continuous and integrable h we conclude from $\lim\limits_{t \downarrow 0} u(x, t) = h(x)$ also

$$h(x) = \lim_{t \downarrow 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i (x-y)k} h(y) d^n y d^n k.$$

We define the Fourier transform of h as $\quad \hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k y} h(y) d^n y. \quad$ This gives

$$u(x, t) = \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i k x} \hat{h}(k) d^n k \quad \text{and} \quad h(x) = \lim_{t \downarrow 0} \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i k x} \hat{h}(k) d^n k.$$

Definition 4.19. *The Schwartz space $\mathcal{S}$ contains all smooth complex valued functions f on $\mathbb{R}^n$ whose functions $x \mapsto |x|^{2l} |\partial^\alpha f(x)|$ are bounded for all $l \in \mathbb{N}$ and all $\alpha \in \mathbb{N}_0^n$.*

Lemma 4.20. *The Fourier transformation maps $\mathcal{S}$ onto $\mathcal{S}$. The inverse is given by*

$$\mathsf{P} \circ \mathcal{F} : \mathcal{S} \to \mathcal{S}, \quad \hat{h} \mapsto h, \quad \text{with} \quad h(x) = \int_{\mathbb{R}^n} e^{2\pi i k x} \hat{h}(k) d^n k.$$

Proof. By two partial integrations we calculate

$$-\widehat{\triangle h}(k) = -\int_{\mathbb{R}^n} e^{-2\pi i k y} \triangle h(y) d^n y = \int_{\mathbb{R}^n} 4\pi^2 k^2 e^{-2\pi i k y} h(y) d^n y = 4\pi^2 k^2 \hat{h}(k).$$

So by $|\hat{h}(k)| \le \int_{\mathbb{R}^n} |h(y)| d^n y$ the Fourier transform of any Schwartz function decays faster then every inverse power of the coordinate. For any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ we obtain

$$\|\hat{h}\|_\infty \le \|h\|_{L^1(\mathbb{R}^n)}.$$

Since $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is dense in $L^1(\mathbb{R}^n)$, the Fourier transform extends to a continuous linear map from $L^1(\mathbb{R}^n)$ into the Banach space $C_b(\mathbb{R}^n, \mathbb{C})$. Furthermore, we have

$$|\partial_i \hat{h}(k)| = \left| \int_{\mathbb{R}^n} -2\pi i y_i e^{-2\pi i y k} h(y) d^n y \right| \le 2\pi \| |y| h(y) \|_{L^1(\mathbb{R}^n)}.$$

So $\hat{h}$ is smooth, if h decays faster than every inverse power of the coordinate. So the Fourier transform of an integrable function is continuous and the Fourier transform of a Schwartz function is smooth.

Theorem 4.3 implies for any $h \in \mathcal{S}$

$$h(x) = \lim_{t \downarrow 0} \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i k x} \hat{h}(k) d^n k \quad \text{with} \quad \hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k y} h(y) d^n y.$$

Since $e^{-4\pi^2 k^2 t}$ converges in the limit $t \downarrow 0$ on any compact subset $K \subset \mathbb{R}^n$ uniformly to 1 and since $\hat{h} \in \mathcal{S}$ belongs to $L^1(\mathbb{R}^n)$, we also have $P \circ \mathcal{F} \circ \mathcal{F} = \mathbb{1}_{\mathcal{S}}$ and $\mathcal{F} \circ \mathcal{F} = \mathsf{P}$, respectively. Now the equation

$$\int_{\mathbb{R}^n} e^{2\pi i k x} \hat{h}(k) d^n k = \int_{\mathbb{R}^n} e^{-2\pi i k x} \hat{h}(-k) d^n k$$

implies $\mathsf{P} \circ \mathcal{F} = \mathcal{F} \circ \mathsf{P}$ and therefore also $\mathcal{F} \circ \mathsf{P} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F} \circ \mathsf{P} = \mathbb{1}_{\mathcal{S}}$. **q.e.d.**

For any Schwartz functions h and g we apply Fubini's Theorem and obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{h}(k)\bar{\hat{g}}(k) d^n k &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{h}(k) \bar{g}(y) e^{2\pi i k y} d^n y d^n k \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{h}(k) e^{2\pi i k y} \bar{g}(y) d^n k d^n y = \int_{\mathbb{R}^n} h(y) \bar{g}(y) d^n y.\end{aligned}$$

This shows that the Fourier transform preserves the hermitian scalar product and the $L^2(\mathbb{R}^n)$-norm. Since the Schwartz space is dense in $L^2(\mathbb{R}^n)$ this implies that the Fourier transform extends to an unitary operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Definition 4.21. *For any open connected domain $\Omega \subset \mathbb{R}^n$ let $W^{2,2}_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in the Hilbert space with the scalar product*

$$\langle f, g \rangle_{W^{2,2}_0(\Omega)} = \int_\Omega (\triangle f) \triangle \bar{g} d^n x + \int_\Omega f \bar{g} d^n x.$$

All functions $h \in C^\infty_0(\Omega)$ obey

$$\langle \triangle h, \triangle h \rangle_{L^2(\Omega)} = \int_\Omega (\triangle h) \triangle \bar{h} d^n x \leq \langle h, h \rangle_{W^{2,2}_0(\Omega)}.$$

Therefore for any $h \in W^{2,2}_0(\Omega)$ the function $\triangle h$ belongs to $L^2(\Omega)$. For $f \in L^2(\Omega)$ the Cauchy Schwarz inequality implies

$$\left| \langle f, \triangle h \rangle_{L^2(\Omega)} \right| \leq \|f\|_{L^2(\Omega)} \cdot \|h\|_{W^{2,2}_0(\Omega)}.$$

A sequence $(h_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ converges together with $(\triangle h_n)_{n\in\mathbb{N}}$ in $L^2(\Omega)$, if and only if it converges in $W^{2,2}(\Omega)$. So the operator $H = -\triangle$ is a closed self adjoint operator on $L^2(\Omega)$ with domain $W_0^{2,2}(\Omega) \subset L^2(\Omega)$. By the inequality

$$\int_\Omega (-\triangle h)\bar{h} d^n x = \int_\Omega |\nabla h|^2 \geq 0 \quad \text{for all} \quad h \in C_0^\infty(\Omega),$$

H is non negative. Hence the operator H has a spectral decomposition and e^{-tH} is a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$ such that the following equation holds:

$$\|e^{-tH}h\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)}.$$

This shows that $u(x,t) = (e^{-tH}h)(x)$ solves $\dot{u}(x,t) = -(He^{-tH})(x) = \triangle u(x,t)$ with Dirichlet boundary condition

$$u(x,0) = h(x) \qquad\qquad u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega.$$

We shall calculate with the help of this relation between the spectral theory of the Laplace operator with Dirichlet boundary condition and the heat equation the heat kernel of the circle S^1 and the interval $[-1,1]$.

4.7 Heat Kernel of S^1

We identify the circle S^1 with the quotient $\mathbb{R}/\mathbb{Z}$. The eigenfunctions of $-\frac{d^2}{dx^2}$ on $\mathbb{R}/\mathbb{Z}$ are equal to $e^{2\pi ikx}$ with $k \in \mathbb{Z}$ with eigenvalues $4\pi^2k^2$. This eigenfunctions build an orthogonal system of the Hilbert space $L^2(\mathbb{R}/\mathbb{Z})$. By the Theorem of Stone and Weierstraß the algebra of polynomials with respect to $\sin(2\pi x)$ and $\cos(2\pi x)$ are dense in the real Banach space $C(\mathbb{R}/\mathbb{Z},\mathbb{R})$. This in turn implies that the same holds for polynomials with respect to $e^{2\pi\imath x}$ and $e^{-2\pi\imath x}$ in the complex Banach space $C(\mathbb{R}/\mathbb{Z},\mathbb{C})$. Therefore the orthogonal complement in $L^2(\mathbb{R}/\mathbb{Z})$ of the former orthogonal system is trivial, and this system is an orthogonal basis. So any $h \in L^2(\mathbb{R}/\mathbb{Z})$ may be decomposed into a series of the eigenfunctions $e^{2\pi ikx}$ of $-\frac{d^2}{dx^2}$ on $\mathbb{R}/\mathbb{Z}$ with eigenvalues $4\pi^2k^2$:

$$h(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx} \qquad \text{with} \qquad a_k = \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi iky} h(y)dy.$$

Therefore the heat kernel of $\mathbb{R}/\mathbb{Z}$ is given by

$$H_{\mathbb{R}/\mathbb{Z}}(x,y,t) = \sum_{k\in\mathbb{Z}} e^{-4\pi^2k^2t+2\pi ik(x-y)} = \Theta(x-y, 4\pi it) \quad \text{with} \ \Theta(x,\tau) = \sum_{k\in\mathbb{Z}} e^{2\pi ikx+\pi i\tau k^2}.$$

Here $\Theta(x,\tau)$ is Jacobi's Theta function. This sum converges on the domain $(x,\tau) \in \mathbb{C} \times \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ to a holomorphic functions since $e^{\pi i \tau k^2}$ decays exponentially with respect to k^2. This Theta function is characterised by the following properties:

$$\Theta(x+1,\tau) = \Theta(x,\tau), \qquad \Theta(x+\tau,\tau) = \Theta(x,\tau)e^{-\pi i\tau - 2\pi i x}, \qquad \Theta(\tfrac{1}{2}+\tfrac{1}{2}\tau,\tau) = 0.$$

The first property follows from the periodicity of $e^{2\pi i k x}$ with period 1. The other two properties we show by direct calculation:

$$\begin{aligned}
\Theta(x+\tau,\tau) &= \sum_{k\in\mathbb{Z}} e^{2\pi i k(x+\tau)+\pi i k^2\tau} & &= \sum_{k\in\mathbb{Z}} e^{2\pi i k x + \pi i (k+1)^2\tau - \pi i \tau} \\
&= \sum_{k\in\mathbb{Z}} e^{2\pi i (k+1)x + \pi i (k+1)^2\tau - 2\pi i x - \pi i\tau} & &= \Theta(x,\tau)e^{-2\pi i x - \pi i \tau} \\
\Theta(\tfrac{1}{2}+\tfrac{\tau}{2},t) &= \sum_{k\in\mathbb{Z}} (-1)^k e^{\pi i\tau((k+\frac{1}{2})^2-\frac{1}{4})} & &= e^{-\frac{4\pi i\tau}{4}} \sum_{l\in\mathbb{N}_0} e^{\pi i \tau(l+\frac{1}{2})^2}(1-1) = 0.
\end{aligned}$$

Exercise 4.22. **(i)** *Show that for all $t > 0$ the fundamental solution $\Phi(x,t)$ belongs to the Schwartz space considered as a function on $x \in \mathbb{R}^n$.*

(ii) *Calculate for all $t > 0$ the Fourier transform of the fundamental solution $\Phi(x,t)$ considered as a function on $x \in \mathbb{R}^n$.*

(iii) *Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth function $\tilde{f}$ on $\mathbb{R}$ which is periodic with period 1:*

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(x+n).$$

(iv) *Let h be a periodic continuous functions on $\mathbb{R}$ with period 1. Show that the solution of the heat equation with initial values h preserves periodicity with period 1 for all $t > 0$. Conclude that the following series is the heat kernel of S^1:*

$$\sum_{n\in\mathbb{Z}} \Phi(x-y+n,t).$$

(v) *Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies*

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n x}.$$

Show with the help of this formula the relation

$$H_{S^1}(x,y,t) = \sum_{n\in\mathbb{Z}} \Phi(x-y+n,t).$$

(vi) *Show that $f(x) = e^{-x^2}(e^{-x^2} + \sin^2 x)$ is a positive Schwartz function on $\mathbb{R}$, whose square root does not belong to the Schwartz space.*

4.8 Heat Kernel of $[0,1]$

The eigenfunctions of $-\frac{d^2}{dx^2}$ on $[0,1]$ with Dirichlet boundary conditions, this means roots at $\partial[0,1]=\{0,1\}$, are given by

$$\sqrt{2}\sin(k\pi x) \quad \text{with} \quad k\in\mathbb{N}$$

These functions again build an orthogonal system:

$$\int_0^1 \sqrt{2}\sin(k\pi x)\sqrt{2}\sin(k'\pi x)dx = \int_0^1 (\cos((k-k')\pi x)-\cos((k+k')\pi x))dx = \delta_{k,k'}.$$

For any continuous functions f on $[0,1]$ with roots at $\partial[0,1]$ the function

$$\tilde{f}(x)=\begin{cases} f(x-2n) & \text{for } x\in[2n,2n+1] \text{ with } n\in\mathbb{Z}\\ -f(2n-x) & \text{for } x\in[2n-1,2n] \text{ with } n\in\mathbb{Z}\end{cases}$$

is continuous on $\mathbb{R}$ with roots at $\mathbb{Z}$ and is periodic with period 2. By the Theorem of Stone and Weierstraß the finite linear combinations of $(x\mapsto\exp(k\pi\imath x))_{k\in\mathbb{N}}$ build a dense subalgebra of $C(\mathbb{R}/2\mathbb{Z})$ and therefore are also dense in $L^2(\mathbb{R}/2\mathbb{Z})$. The map $f\mapsto\tilde{f}$ obeys $\int_0^2\tilde{f}(x)\tilde{g}(x)dx=2\int_0^1 f(x)g(x)dx$ and maps $L^2[0,1]$ onto

$$\mathcal{A}=\left\{f\in L^2(\mathbb{R}/2\mathbb{Z})\;\middle|\; f(n+x)=\begin{cases} f(x) & \text{for even } n\in 2\mathbb{Z} \text{ and } x\in\mathbb{R}\\ -f(1-x) & \text{for odd } n\in 2\mathbb{Z}+1 \text{ and } x\in\mathbb{R}\end{cases}\right\}$$

This space $\mathcal{A}$ consists of all periodic odd functions on $\mathbb{R}$ with period 2, since $n=-1$ gives $f(x-1)=-f(1-x)$. A linear combination $\sum_k a_k\exp(k\pi\imath x)$ belongs to $\mathcal{A}$, if and only if $a_{-k}=-a_k$ for all $k\in\mathbb{Z}$. Hence the linear combinations of $(\sqrt{2}\sin(k\pi x))_{k\in\mathbb{N}}$ are dense in $\mathcal{A}$ and build an orthonormal basis of $L^2[0,1]$. This implies

$$h=\sum_{k\in\mathbb{N}} a_k\sqrt{2}\sin(k\pi x) \quad \text{with} \quad a_k=\int_0^1\sqrt{2}\sin(k\pi y)h(y)dy \quad \text{for } h\in L^2[0,1].$$

We conclude that the unique solution of the initial value problem

$$\dot{u}(x,t)-\triangle u(x,t)=0 \quad u(x,0)=h(x) \quad u(0,t)=u(1,t)=0 \quad \text{for } (x,t)\in(0,1)\times\mathbb{R}^+$$

is given by
$$u(x,t)=\int_0^1 H_{[0,1]}(x,y,t)h(y)dy \qquad \text{with}$$

$$\begin{aligned} H_{[0,1]}(x,y,t) &= \sum_{k=1}^{\infty} e^{-\pi^2k^2t}2\sin(k\pi x)\sin(k\pi y)\\ &= \sum_{k=1}^{\infty} e^{-\pi^2k^2t}(\cos(k\pi(x-y))-\cos(k\pi(x+y))) = \frac{1}{2}\Theta(\tfrac{x-y}{2},\pi it)-\frac{1}{2}\Theta(\tfrac{x+y}{2},\pi it).\end{aligned}$$

Exercise 4.23. **(i)** *Show that the heat kernel $H_{[0,1]}$ is given by*

$$H_{[0,1]}(x, y, t) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi\imath t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi\imath t).$$

(ii) *Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:*

$$f(n + x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1 - x) & \text{for odd } n \in 2\mathbb{Z} + 1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2.

(iii) *Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth functions $\tilde{f}$ in $\mathcal{A}$:*

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(2n + x) - \sum_{n\in\mathbb{Z}} f(2n - x).$$

(iv) *Show for any $h \in \mathcal{A}$, that the solutions of the heat equation with initial value h is for all $t > 0$ a smooth function in $\mathcal{A}$. Conclude from this that the following sum has the properties of the Heat kernel of $[0, 1]$:*

$$\sum_{n\in\mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n\in\mathbb{Z}} \Phi(x + 2n + y, t).$$

(v) *Show the relation*

$$H_{[0,1]}(x, y, t) = \sum_{n\in\mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n\in\mathbb{Z}} \Phi(x + 2n + y, t).$$

The heat kernel of the Cartesian product of two domains can be easily calculated in terms of the heat kernels of both domains:

Lemma 4.24. *If $\Omega \subset \mathbb{R}^m$ and $\Omega' \subset \mathbb{R}^n$ are two open, bounded and connected domains with given heat kernels H_Ω and $H_{\Omega'}$, then the heat kernel of $\Omega \times \Omega'$ is given by*

$$H_{\Omega\times\Omega'}((x, x'), (y, y'), t) = H_\Omega(x, y, t)H_{\Omega'}(x', y', t) \quad (x, x'), (y, y') \in \bar{\Omega} \times \bar{\Omega}' \quad t \in \mathbb{R}^+.$$

Proof. For any $(x, x', t) \in \Omega \times \Omega' \times \mathbb{R}^+$ the function $(y, y') \mapsto H_\Omega(x, y, t)H_{\Omega'}(x', y', t)$ extends by the value zero continuously to $\partial(\Omega \times \Omega') = (\partial\Omega \times \Omega') \cup (\omega \times \partial\Omega')$. The Laplace operator of the Cartesian product is the sum of the corresponding Laplace operators. Hence for all $(x,x') \in \Omega \times \Omega'$ the function $(y,y',t) \mapsto H_\Omega(x,y,t)H_{\Omega'}(x'y',t)$ solves the homogeneous heat equation. The product of both fundamental solutions is the fundamental solution on $\mathbb{R}^{m+n}$. Hence for all $(x,x') \in \Omega \times \Omega'$ the function $(y,y',t) \mapsto H_\Omega(x,y,t)H_{\Omega'}(x',y't) - \Phi(x-y,t)\Phi(x'-y',t)$ extends continuously to $\bar{\Omega} \times \bar{\Omega}' \times \mathbb{R}_0^+$ by setting it zero on $(y,y',t) \in \bar{\Omega} \times \bar{\Omega}' \times \{0\}$. **q.e.d.**

So we might have a formula for the heat kernels all tori $(\mathbb{R}/\mathbb{Z})^n$ and all Cartesian products $[0,1]^n$. However the boundaries of the Cartesian products $[0,1]^n \subset \mathbb{R}^n$ are no continuously differentiable submanifolds of $\mathbb{R}^n$ and our proof of the divergence theorem does not apply to these Cartesian products. However, the divergence theorem holds for these Cartesian products and we prove this in the lecture Partial Differential Equations. So we have determined the heat kernel of all tori $(\mathbb{R}/\mathbb{Z})^n$ and all Cartesian products $[0,1]^n$. Hence the unique solution of the initial value problem

$$\dot{u} - \triangle u = 0 \text{ on } (0,1)^n \times (0,T], \quad u(x,0) = h(x) \text{ on } [0,1]^n, \quad u = 0 \text{ on } \partial[0,1]^n \times [0,T]$$

is given by $\qquad u(x,t) = \int_{[0,1]^n} \prod_{i=1}^n H_{[0,1]}(x_i,y_i,t)h(y)d^ny.$

From $\Phi(x-y,t) = \frac{1}{r^n}\Phi(\frac{x}{r}-\frac{y}{r},\frac{t}{r^2})$ we obtain $H_{[0,r]^n}(x,y,t) = \frac{1}{r^n}\prod_{i=1}^n H_{[0,1]}\left(\frac{x_i}{r},\frac{y_i}{r},\frac{t}{r^2}\right)$.

Corollary 4.25. *Any solution $u(x,t)$ of the homogeneous heat equation on a neighbourhood of $[0,r]^n \times [0,T] \subset \mathbb{R}^n \times \mathbb{R}$ satisfies*

$$u(x,T) = -\int_0^T\!\!\int_{\partial[0,r]^n} u(z,t)\nabla_z H_{[0,r]^n}(x,z,T-t)N(z)d\sigma(z)dt + \int_{[0,r]^n} u(y,0)H_{[0,r]^n}(x,y,T)d^ny.$$

q.e.d.

In the proof of Theorem 4.3 we show that in the limit $t \downarrow 0$ $\Phi(x-y,t)$ converges on the complement of $y \in B(x,\delta)$ uniformly to zero. The same is true for all partial derivatives and due to condition (ii) in Definition 4.14 also for $H_{[0,1]^n}(x,y,t)$. By Lemma 4.17 the integral for $u(x,T)$ is smooth at all $x \in (0,r)^n$. For $(z,t) \in \partial[0,r]^n \times [0,T]$ the Taylor series of $x \mapsto H_{[0,r]^n}(x,z,T-t)$ converges uniformly on compact subsets of $x \in (0,r)^n$ to $H_{[0,r]^n}(x,z,T-t)$. This implies the following Corollary:

Corollary 4.26. *Any solution u of the homogeneous heat equation on an open domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is smooth and for fixed t analytic with respect to x.* **q.e.d.**