# Chapter 3

# Laplace Equation

One of the most important PDEs is the Laplace equation

$$\triangle u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

The corresponding inhomogeneous PDE is Poisson's equation

$$-\triangle u = f.$$

Both equations are linear PDEs of second order with the unknown function $u : \mathbb{R}^n \to \mathbb{R}$. They show up in many situations. In physics they describe for example the potential of an electric field in the vacuum with some distribution of charges $f$.

## 3.1   Fundamental Solution

The Laplace equation is invariant with respect to all rotations and translations of the Euclidean space $\mathbb{R}^n$. Therefore we first look for solutions which are invariant with respect to all rotations. These solutions depend only on the length $r = |x| = \sqrt{x \cdot x}$ of the position vector $x$. For such functions $u(x) = v(r) = v(\sqrt{x \cdot x})$ we calculate:

$$\nabla_x u(x) = v'\left(\sqrt{x \cdot x}\right) \nabla_x r = v'\left(\sqrt{x \cdot x}\right) \frac{2x}{2r}.$$

Hence the Laplace equation simplifies to an ODE

$$\triangle_x u(x) = \nabla_x \cdot \nabla_x u = v''(r)\frac{x^2}{r^2} + v'(r)\frac{n}{r} - v'(r)\frac{x^2}{r^2 r} = v''(r) + \frac{n-1}{r}v'(r) = 0.$$

Let us solve this ODE:

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \;\Rightarrow\; \ln(v'(r)) = (1-n)\ln(r) + C \;\Rightarrow\; v(r) = \begin{cases} C' \ln(r) + C'' & \text{for } n = 2 \\ \frac{C'}{r^{n-2}} + C'' & \text{for } n \geq 3. \end{cases}$$

**Definition 3.1.** *Let $\Phi(x)$ be the following solutions of the Laplace equation:*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & \text{for } n = 2 \\ \frac{1}{n(n-2)\omega_n|x|^{n-2}} & \text{for } n \geq 3. \end{cases}$$

*Here $\omega_n$ denotes the volume of the unit ball $B(0,1)$ in Euclidean space $\mathbb{R}^n$.*

This solution has a singularity at the origin $x = 0$. We have chosen the constants in such a way that the following theorem holds:

**Theorem 3.2.** *For $f \in C_0^2(\mathbb{R}^n)$ a solution of Poisson's equations $-\triangle u = f$ is given by*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)d^ny = \int_{\mathbb{R}^n} \Phi(z)f(x-z)d^nz.$$

*Proof.* The equality of both integrals in the definition of $u(x)$ follows from the substitution $z = x - y$. The second integral is twice continuously differentiable, since $f$ is twice continuously differentiable and has compact support. We calculate

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)d^ny.$$

In particular, $\triangle u(x) = \int_{\mathbb{R}^n} \Phi(y)\triangle_x f(x-y)d^ny$. We decompose this integral in the sum of an integral nearby the singularity of $\Phi$ and an integral away from this singularity:

$$\triangle u(x) = \int_{B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)d^ny \qquad + \int_{\mathbb{R}^n\backslash B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)d^ny$$
$$= I_\epsilon \qquad\qquad\qquad\qquad\qquad +J_\epsilon.$$

We use $\int r\ln r\, dr = \frac{r^2}{2}(\ln r - \frac{1}{2})$ and $\int r\, dr = \frac{r^2}{2}$ and estimate the first integral for $\epsilon \downarrow 0$:

$$|I_\epsilon| \leq C\|\triangle_x f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)|d^ny \leq \begin{cases} C\epsilon^2(|\ln\epsilon| + 1) & (n = 2) \\ C\epsilon^2 & (n \geq 3). \end{cases}$$

Integration by parts of the second integral yields

$$J_\epsilon = \int_{\mathbb{R}^n\backslash B(0,\epsilon)} \Phi(y)\triangle_y f(x-y)d^ny$$
$$= -\int_{\mathbb{R}^n\backslash B(0,\epsilon)} \nabla_y\Phi(y)\cdot\nabla_y f(x-y)d^ny \quad + \int_{\partial B(0,\epsilon)} \Phi(y)\nabla_y f(x-y)\cdot N d\sigma(y)$$
$$= K_\epsilon \qquad\qquad\qquad\qquad\qquad +L_\epsilon.$$

Here $N$ is the outer normal and $d\sigma$ the measure on the boundary of $\mathbb{R}^n \setminus B(0, \epsilon)$. The second term converges in the limit $\epsilon \downarrow 0$ to zero:

$$|L_\epsilon| \le |\nabla f|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)| d\sigma(y) \le \begin{cases} C\epsilon |\ln \epsilon| & (n = 2) \\ C\epsilon & (n \ge 3). \end{cases}$$

Another integration by parts of the first term yields

$$\begin{aligned} K_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \triangle_y \Phi(y) f(x - y) d^n y - \int_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x - y) \cdot N d\sigma(y) \\ &= -\int_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x - y) \cdot N d\sigma(y). \end{aligned}$$

Here we used that $\phi$ is harmonic for $y \ne 0$. The gradient of $\Phi$ is equal to $\nabla \Phi(y) = -\frac{1}{n\omega_n} \frac{y}{|y|^n}$. The outer normal points towards the origin and is equal to $-\frac{y}{|y|}$. Let $\sigma_n(r)$ denote the area of $\partial B(0, r) \subset \mathbb{R}^n$. By the divergence theorem for $x \mapsto x$ we have

$$n\omega_n r^n = \int_{B(0,r)} \nabla \cdot x d\mu = \int_{\partial B(0,r)} x \cdot N(x) d\sigma(x) = \int_{\partial B(0,r)} x \cdot \frac{x}{|x|} d\sigma(x) = r\sigma_n(r), \quad \sigma_n(r) = n\omega_n r^{n-1}.$$

Now $K_\epsilon$ is the mean value of $-f$ on $\partial B(0, \epsilon)$, since $\sigma_n(\epsilon) = n\omega_n \epsilon^{n-1}$ is the area of $\partial B(0, \epsilon)$. By continuity of $f$ this mean value converges for $\epsilon \downarrow 0$ to $-f(x)$. **q.e.d.**

By this theorem we have $-\triangle \Phi(x) = \delta(x)$ in the sense of distributions. This relation justifies the choice of the constant in the definition of $\phi$. The convolution of $f$ with $\Phi$ is defined for continuous functions $f \in L^1(\mathbb{R}^n)$. On can show that as a measurable function the convolution is defined even for $f \in L^1(\mathbb{R}^n)$ and belongs to $L^1(\mathbb{R}^n)$. In this case the convolution is in the sense of distributions a solution of Poisson's equation. In general, for a continuous $f \in L^1(\mathbb{R}^n)$ the corresponding $u$ is not twice differentiable, and the theorem is not valid for such $f$. But it is valid for all Lipschitz continuous functions $f$ in $L^1(\mathbb{R}^n)$. Since Poisson's equation is an inhomogeneous linear PDE, all solutions are defined up to adding a solution of the homogeneous equation which is Laplace equation.

## 3.2 Mean Value Property

In this section we shall prove the following property of a harmonic function $u$ on an open domain $\Omega \subset \mathbb{R}^n$: the value $u(x)$ of $u$ at the center of any ball $B(x, r)$ with compact closure in $\Omega$ is equal to the mean of $u$ on the boundary of the ball. Conversely, if this holds for all balls with compact closure in $\Omega$, then $u$ is harmonic. The mean of $u$ on

the ball $B(x, r)$ is the mean over $r' \in [0, r]$ of the means of $u$ on the boundary of $B(x, r')$. Therefore the same statement holds for the means of $u$ on the balls $B(x, r)$. This relation is called mean value property and has many important consequences.

**Mean Value Property 3.3.** *Let $u \in C^2(\Omega)$ be harmonic on an open domain $\Omega \subset \mathbb{R}^n$ containing the closure of the ball $B(x, r)$. The mean of $u$ on the ball $B(x, r)$ and on its boundary is equal to the value $u(x)$ of $u$ at the center. Conversely, if the means of a function $u \in C^2(\Omega)$ on all balls $B(x, r)$ with compact closure in $\Omega$ or on all boundaries of such balls is equal to the value $u(x)$ at the center of the ball, then $u$ is harmonic.*

*Proof.* We define $\Phi(r)$ for $x \in \Omega$ as the mean of $u$ on $\partial B(x, r) \subseteq \Omega$:

$$\Phi(r) := \frac{1}{r^{n-1} n \omega_n} \int_{\partial B(x,r)} u(y) \, d\sigma(y) = \frac{1}{n \omega_n} \int_{\partial B(0,1)} u(x + rz) \, d\sigma(z).$$

Here $\omega_n$ denotes the volume of the unit ball in Euclidean space $\mathbb{R}^n$. We apply the Divergence Theorem and calculate the derivative $\qquad \Phi'(r) =$

$$= \frac{1}{n \omega_n} \int_{\partial B(0,1)} \nabla u(x + rz) \cdot z \, d\sigma(z) = \frac{1}{r^{n-1} n \omega_n} \int_{\partial B(x,r)} \nabla u(y) \cdot N \, d\sigma(y) = \frac{1}{r^{n-1} n \omega_n} \int_{B(x,r)} \triangle u \, d\mu.$$

Hence for harmonic $u$ this function is constant as long as $B(x, r)$ has compact closure in $\Omega$. By continuity of $u$ this function $\Phi(r)$ converges in the limit $\lim r \to 0$ to $u(x)$. This shows that the means of $u$ on all spheres $\partial B(x, r)$ are equal to the values $u(x)$ of $u$ at the center $x$. Now we claim that the integral of the function $u$ over $B(x, r)$ is

$$\int_{B(x,r)} u(y) d\mu(y) = \int_{B(0,r)} u(x + y) d\mu(y) = \int_0^r \int_{\partial B(0,s)} u(x + y) d\sigma(y) ds = \int_0^r \int_{\partial B(x,s)} u(y) d\sigma(y) ds.$$

In fact for any linear rotation $\mathsf{O}$ the square of the determinant of the Jacobean of the map $(z, r) \mapsto \mathsf{O}(z, \sqrt{r^2 - z^2})$ is equal to $\frac{r^2}{r^2 - z^2}$ which is equal to $\det\left((\Phi'(z))^T \Phi'(z)\right) = 1 + (\nabla_z \sqrt{r^2 - z^2})^2$ of the pasteurisation $z \mapsto \Phi(z) = \mathsf{O}(z, \sqrt{r^2 - z^2})$ of $\partial B(0, r)$. Hence the claim follows from Jacobi's transformation formula and the Definition 2.3 of the integral over $\partial B(0, r)$. So the mean of $u$ on the ball $B(x, r)$ is equal to

$$\frac{1}{r^n \omega_n} \int_{B(x,r)} u(y) \, d^n y = \frac{n}{r^n} \int_0^r \frac{1}{s^{n-1} n \omega_n} \int_{\partial B(x,s)} s^{n-1} u(y) \, d\sigma(y) \, ds = \frac{n}{r^n} \int_0^r s^{n-1} \Phi(s) \, ds.$$

For constant $\Phi$ this is again equal to the value $u(x)$ of $u$ at the center $x$.

Conversely, if the means of $u$ on all balls $B(x, r)$ with compact closure in $\Omega$ is equal to the values $u(x)$ of $u$ at the center $x$, then we have

$$u(x) = \frac{1}{\omega_n r^n} \int_0^r n \omega_n s^{n-1} \Phi(s) \, ds = \frac{n}{r^n} \int_0^r s^{n-1} \Phi(s) \, ds.$$

The vanishing of the derivative with respect to $r$ of the right hand side yields

$$0 = -\frac{n^2}{r^{n+1}} \int_0^r s^{n-1} \Phi(s) \, ds + \frac{n}{r^n} r^{n-1} \Phi(r) = -\frac{n}{r} u(x) + \frac{n}{r} \Phi(r).$$

Therefore also the means $\Phi(r)$ of $u$ on the boundaries $\partial B(x, r)$ are equal to the value $u(x)$ of $u$ at the center $x$. Since $u$ is twice continuously differentiable, the function $\Phi$ is twice continuously differentiable. We have seen that $\Phi'(r)$ is the mean of $\triangle u$ on $B(x, r)$. In particular, if the mean of $u$ on all $\partial B(x, r) \subset \Omega$ is equal to the value $u(x)$ of $u$ at the center $x$, then the integral of $\triangle u$ over the balls with compact closure in $\Omega$ vanish. If there exists $x \in \Omega$ with $\triangle u(x) \neq 0$, then there is a ball $B(x, r)$ with compact closure in $\Omega$, such that either $\triangle u < -|\triangle u(x)|/2$ or $\triangle u > |\triangle u(x)|/2$ on $B(x, r)$ and $\int_{B(x,r)} \triangle u(y) d^n y \neq 0$. Therefore $\triangle u$ vanishes and $u$ is harmonic on $\Omega$.          **q.e.d.**

**Corollary 3.4.** *Let $u$ be a smooth[1] harmonic function on an open domain $\Omega \subset \mathbb{R}^n$ and $B(x, r)$ a ball with compact closure in $\Omega$. For all multi-indices $\alpha$ we have the estimate*

$$|\partial^\alpha u(x)| \leq C(n, |\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(\overline{B(x,r)})} \quad \text{with} \quad C(n, |\alpha|) = 2^{\frac{|\alpha|(1+|\alpha|)}{2}} n^{|\alpha|}.$$

*Proof.* All partial derivatives of a harmonic function are harmonic. The Mean Value Property and the Divergence Theorem yield for $i = 1, \ldots, n$

$$|\partial_i \partial^\alpha u(x)| = \left| \frac{2^n}{\omega_n r^n} \int_{B(x,r/2)} \partial_i \partial^\alpha u \, d\mu \right| = \left| \frac{2^n}{\omega_n r^n} \int_{\partial B(x,r/2)} \partial^\alpha u N_i d\sigma \right| \leq \frac{2n}{r} \|\partial^\alpha u\|_{L^\infty(\partial B(x,r/2))}.$$

The inductive application gives first $C(n, 1) = 2n$, and using the induction hypothesis

$$\|\partial^\alpha u(y)\| \leq 2^{|\alpha|} C(n, |\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(B(x,r))} \quad \text{for all} \quad y \in \partial B(x, r/2)$$

the relation $C(n, 1+|\alpha|) = 2^{1+|\alpha|} n C(n, |\alpha|)$. The given $C(n, |\alpha|)$ is the solution. **q.e.d.**

**Liouville's Theorem 3.5.** *On $\mathbb{R}^n$ a bounded harmonic function is constant.*

*Proof.* The foregoing corollary shows that $|\partial_i u(x)|$ is bounded by $2n \|u\|_{L^\infty(\mathbb{R}^n)} r^{-1}$ for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$. In the limit $r \to \infty$ the first partial derivatives vanish identically. Therefore $u$ is constant.          **q.e.d.**

Let us now transfer the Mean Value Property to a property of distributions. If $u \in C^2(\Omega)$ is harmonic, then for $B(x, r) \subset \Omega$ and $\psi \in C_0^\infty((0, r))$ we have

$$\int_{B(x,r)} u(y) \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} \, d^n y = \int_0^r \frac{\psi(s)}{ns^{n-1}\omega_n} \int_{\partial B(x,s)} u(y) \, d\sigma(y) \, ds = \left( \int_0^r \psi(s) \, ds \right) u(x).$$

So the distribution $F_u$ has the following property:

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[1]We shall see in Weyl's Lemma that this assumption can be replaced by the assumption $u \in L^\infty_{\text{loc}}(\Omega)$.

**Weak Mean Value Property 3.6.** *Let $U \in \mathcal{D}'(\Omega)$ be a harmonic distribution on an open domain $\Omega \subset \mathbb{R}^n$. For each ball $B(x, r)$ with $\overline{B}(x, r) \subset \Omega$ and each $\psi \in C_0^\infty((0, r))$ with $\int \psi \, d\mu = 0$ the distribution $U$ vanishes on the following test function:*

$$f \in C_0^\infty(\Omega), \quad y \mapsto f(y) = \frac{\psi(|y - x|)}{n|y - x|^{n-1}\omega_n} \quad with \quad \operatorname{supp} f \subset B(x, r) \subset \Omega.$$

*Proof.* It suffices to show that there exists a test function $g \in C_0^\infty(\Omega)$ with $\triangle g = f$. By the assumptions on $\psi$ there exists a test function $\Psi \in C_0^\infty((0, r))$ with $\Psi' = \psi$. We define

$$g(y) = v(|y - x|) \quad \text{with} \quad v(t) = \int_r^t \frac{\Psi(s)}{ns^{n-1}\omega_n} \, ds.$$

This function $g$ has compact support in $B(x, r) \subset \Omega$, depends only on $|y - x|$ and is constant on $B(x, \epsilon)$ for some $\epsilon > 0$. We calculate for $y \neq x$:

$$\triangle_y g(y) \qquad\qquad = v''(|y - x|) + \frac{n-1}{|y - x|}v'(|y - x|)$$

This implies

$$\triangle_y g(y) = \frac{\psi(|y - x|)}{n|y - x|^{n-1}\omega_n} - \frac{(n-1)\Psi(|y - x|)}{n|y - x|^n\omega_n} + \frac{n-1}{|y - x|}\frac{\Psi(|y - x|)}{n|y - x|^{n-1}\omega_n} = f(y). \quad \textbf{q.e.d.}$$

**Weyl's Lemma 3.7.** *On an open domain $\Omega \subset \mathbb{R}^n$ for each harmonic distribution $U \in \mathcal{D}'(\Omega)$ there exists a harmonic function $u \in C^\infty(\Omega)$ with $U = F_u$.*

*Proof.* Let us first define $u$. For all $x \in \Omega$ choose a ball $B(x, r) \subset \Omega$ and a test function $\psi \in C_0^\infty((0, r))$ with $\int_0^r \psi(s) \, ds = 1$. Then we define

$$u(x) := U(g_x) \quad \text{with} \qquad\qquad g_x(y) := \frac{\psi(|y - x|)}{n|y - x|^{n-1}\omega_n}.$$

If $U$ is harmonic, then the Weak Mean Value Property implies that $u(x)$ does not depend on the choice of $r$ and $\psi$. Hence we can use in the formula for $u(x)$ the same $r$ and $\psi$ for all $x$ in a small neighbourhood of each $x_0$. Then $u$ is the convolution of the test function $g_0 = \mathsf{P}g_0$ with the distribution $U$. Due to Lemma 2.8, $u$ is smooth.

Next we prove that the distribution $\tilde{U} = F_u$ has the Weak Mean Value Property. By the discussion which motivates Lemma 2.8 this distribution $\tilde{U}$ is the convolution of $g_0 = \mathsf{P}g_0$ with $U$, conceived not as the function $u$ but as a distribution $F_u$. The functions $f$ in the Weak Mean Value Property are characterised by three properties:

1. they depend only on the distance $|y - x|$ of the variable $y$ to some center $x \in \Omega$,

2. they vanish on a neighbourhood of the center $x$ and

3. their integrals vanish.

The first property is equivalent to the invariance of $f$ with respect to all rotations around the center. We show that the convolution $g * f$ of a test function $g$ which is invariant with respect to all rotations around the center $x$ with a test function f which is invariant with respect to all rotations around a center $z$ is invariant with respect to all rotations around the center $x + z$. In fact, for all $\mathsf{O} \in O(n, \mathbb{R})$ we use the invariance of the Lebesgue measure with respect to translations and $\mathsf{O}$ and obtain

$$(g*f)(x+z+\mathsf{O}y) = \int_{\mathbb{R}^n} g(x+z+\mathsf{O}y-y')f(y')\,\mathrm{d}^n y' = \int_{\mathbb{R}^n} g(x+\mathsf{O}(y-z'))f(z+\mathsf{O}z')\,\mathrm{d}^n z'.$$

Furthermore, the integral of $g * f$ is the product of the integrals of $g$ and $f$:

$$\int_{\mathbb{R}^n} (g*f)(x)\,\mathrm{d}^n x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^n y\,\mathrm{d}^n x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^n x\,\mathrm{d}^n y$$

$$= \left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}^n x\right)\left(\int_{\mathbb{R}^n} f(y)\,\mathrm{d}^n y\right).$$

In particular, the convolution $g_0 * f$ of the test functions $g_0$ with the test function $f$ in the Weak Mean Value Property is again a test function which is invariant with respect to the rotations around the same center as $f$ with vanishing total integral. Since $\psi$ has compact support in $(0, r)$ the function $f$ vanishes on $B(x, \epsilon)$ for sufficiently small $\epsilon > 0$. If the support of $g_0$ is contained in $B(0, \epsilon/2)$, then $g_0 * f$ vanishes on $B(x, \epsilon/2)$. This implies that $g_0 * f$ is again a function of the form considered in the Weak Mean Value Property and $\tilde{U}(f) = U(g_0 * f) = 0$. So $\tilde{U}$ has the Weak Mean Value Property.

Next we prove that $u$ is harmonic. Let $\phi_{B(0,\epsilon)} \in C_0^\infty(\mathbb{R})$ be the mollifier defined in Section 2.4. For any $B(x, r)$ with compact closure in $\Omega$, there exists $R > r$ with $B(x, R) \subset \Omega$. For all $0 < r_1 < r_2 < R$ and sufficiently small $\epsilon$ the function $\psi(t) = \phi_{B(0,\epsilon)}(t-r_1) - \phi_{B(0,\epsilon)}(t-r_2)$ has compact support in $(0, R)$ and vanishing total integral. Let $f_\epsilon$ denote the corresponding functions in the kernel of $\tilde{U}$. Since $u$ is continuous, the limit $\epsilon \downarrow 0$ of $\tilde{U}(f_\epsilon)$ exists and converges to the difference of the means of $u$ on $\partial B(x, r_2)$ and $\partial B(x, r_1)$. Since $\tilde{U}$ has the Weak Mean Value Property these differences vanish for all $0 < r_1 < r_2 < R$ and the means of $u$ on all $\partial B(x, r) \subset \Omega$ coincide. In the limit $r \downarrow 0$ these means converge to $u(x)$, since $u$ is continuous. Therefore $u$ has the Mean Value Property and is a smooth harmonic function.

Finally we prove $\tilde{U} = U$. The functions $\psi(t) = \phi_{B(0,\epsilon/3)}(t - 2/3\epsilon)$ have support $[\epsilon/3, \epsilon]$ and total integral 1. The corresponding functions $g_0$ are smooth mollifiers $\lambda_\epsilon$. By definition of $\tilde{U}$ we have $\tilde{U} = \lambda_\epsilon * U$. Now Lemma 2.9 implies $\tilde{U} = U$. **q.e.d.**

Actually we have proven that any distribution $U$ that has the Weak Mean Value Property corresponds to a smooth harmonic function. Therefore the weak solutions of the Laplace equations coincide with the strong solutions, and all solutions are smooth.

Let us finish this section with a proof of Harnack's inequality. This inequality estimates the values of a positive harmonic on a path–connected domain in terms of the value at any point in the domain.

**Harnack's Inequality 3.8.** *Let $\Omega'$ be an open path–connected domain with compact closure in the open domain $\Omega \subset \mathbb{R}^n$. Then there exists a constant $C > 0$ depending only on $\Omega'$ and $\Omega$, such that any non-negative harmonic function $u$ on $\Omega$ satisfies the Harnack inequality*

$$\sup_{x \in \Omega'} u(x) \leq C \inf_{x \in \Omega'} u(x).$$

*In particular, for all $x, y \in \Omega'$ we have*
$$\frac{1}{C} u(y) \leq u(x) \leq C u(y).$$

*Proof.* Let $r$ be the minimal value of the continuous function on the compact set $\bar{\Omega}'$:

$$\bar{\Omega}' \to \mathbb{R}^+, \quad x \mapsto \sup\{R > 0 \mid B(x, 2R) \subset \Omega\}.$$

For $x \in \Omega'$ and $y \in B(x, r)$ we have $B(y, r) \subset B(x, 2r) \subset \Omega$. The Mean Value Property implies

$$u(x) = \frac{1}{2^n r^n \omega_n} \int_{B(x,2r)} u \, d\mu \geq \frac{2^{-n}}{r^n \omega_n} \int_{B(y,r)} u \, d\mu = 2^{-n} u(y).$$

Since $\bar{\Omega}'$ is compact and path–connected, it can be covered by finitely many balls $B_1, \ldots, B_N$ of radius $\frac{r}{2}$ such that $B_{n+1} \cap B_n \neq \emptyset$ for $n = 1, \ldots, N-1$. An $N$-fold application of the special case implies for general $x, y \in \Omega'$

$$u(x) \geq 2^{-nN} u(y).$$

Taking the infimum over $x \in \Omega'$ and the supremum over $y \in \Omega'$ gives

$$\sup_{x \in \Omega'} u(x) \leq 2^{nN} \inf_{x \in \Omega'} u(x). \qquad \textbf{q.e.d}.$$

**Harnack's Principle 3.9.** *On an open and path–connected domain $\Omega \subset \mathbb{R}^n$ a monotone sequence of harmonic functions $(u_n)_{n \in \mathbb{N}}$ converges uniformly on all compact subsets if and only if there exists $x \in \Omega$ such that $(|u_n(x)|)_{n \in \mathbb{N}}$ is bounded.*

*Proof.* If $(u_n)_{n \in \mathbb{N}}$ converges uniformly on compact subsets, then $(u_n(x))_{n \in \mathbb{N}}$ converges for all $x \in \Omega$. Conversely, let $(|u_n(x)|)_{n \in \mathbb{N}}$ be bounded for some $x \in \Omega$. By monotonicity the sequence $(u_n(x))_{n \in \mathbb{N}}$ converges. Furthermore, we may assume that $(u_n - u_m)_{n \geq m}$ is non-negative. Harnack's Inequality implies that $(u_n - u_m)_{n \geq m}$ is uniformly bounded on compact subsets of $\Omega$ and monotonic. Hence it converges uniformly there. The limit has together with all $u_n$ the Mean Value Property and is harmonic. **q.e.d.**

## 3.3   Maximum Principle

Let the harmonic function $u$ take in a point $x$ of an open path–connected domain $\Omega \subset \mathbb{R}^n$ a maximum. The Mean Value Property implies on all balls $B(x, r) \subset \Omega$

$$\frac{1}{r^n \omega_n} \int_{B(x,r)} |u(x) - u(y)| d^n y = \frac{1}{r^n \omega_n} \int_{B(x,r)} (u(x) - u(y)) d^n y = 0.$$

Hence $u$ takes the maximum on all these balls $B(x, r) \subset \Omega$. Since $\Omega$ is path–connected every other point $y \in \Omega$ is connected with $x$ by a continuous path $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. The compact image $\gamma[0, 1]$ is covered by finitely many balls $B(\gamma(t_1), r_1), \ldots, B(\gamma(t_N), r_N) \subset \Omega$ with $0 \leq t_1 < \ldots t_N \leq 1$ and $r_1, \ldots, r_N > 0$. Hence $u$ is constant along $\gamma$, and on $\Omega$ since this is true for all $y \in \Omega$. This proves

**Strong Maximum Principle 3.10.** *If a harmonic function $u$ has on a path–connected open domain $\Omega \subset \mathbb{R}^n$ a maximum, then $u$ is constant.* **q.e.d.**

**Weak Maximum Principle 3.11.** *Let the harmonic function $u$ on a bounded open domain $\Omega \subset \mathbb{R}^n$ extend continuously to the boundary $\partial\Omega$. The maximum of $u$ is taken on the boundary $\partial\Omega$.*

*Proof.* By Heine Borel the closure $\bar{\Omega}$ is compact and the continuous function $u$ takes on $\bar{\Omega}$ a maximum. If it does not belong to $\partial\Omega$, then $u$ is constant on the corresponding path–connected component and the maximum is also taken on $\partial\Omega$. **q.e.d.**

Since the negative of a harmonic function is harmonic the same conclusion holds for minima. Now we generalise the Maximum Principle, but not the Mean Value Property.

**Definition 3.12.** *On an open domain $\Omega \subset \mathbb{R}^n$ an differential operator $L$*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i}$$

*with symmetric coefficients $a_{ij} = a_{ji}$ is called elliptic, if*

$$\sum_{i,j=1}^{n} a_{ij}(x) k_i k_j > 0 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n \setminus \{0\}.$$

If we replace $a_{ij}$ by $\frac{1}{2}(a_{ij} + a_{ji})$, then the assumption $a_{ij} = a_{ji}$ is fulfilled. Due to the commutativity of the second derivatives this replacement does not change $L$.

**Theorem 3.13.** *Let $L$ be an elliptic operator on a bounded open domain $\Omega \subset \mathbb{R}^n$ whose coefficients $a_{ij}$ and $b_i$ extend continuously and elliptic to $\partial\Omega$. Every twice differentiable solution $u$ of $Lu \geq 0$ which extends continuously to $\partial\Omega$ takes its maximum on $\partial\Omega$.*

*Proof.* Let us first show that $L$ is uniform elliptic, i.e. there exists $\lambda > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(x)k_i k_j \geq \lambda \sum_{i=1}^{n} k_i^2 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n.$$

The continuous function $(x, k) \mapsto \sum_{i,j=1}^{n} a_{ij}(x)k_i k_j$ attains on the compact set $(x, k) \in \bar{\Omega} \times S^{n-1} \subset \bar{\Omega} \times \mathbb{R}^n$ a minimum $\lambda > 0$. Hence $L$ is uniform elliptic.

For $v(x) = \exp(\alpha x_1)$ with $\alpha > 0$ we conclude

$$Lv = \alpha(\alpha a_{11}(x) + b_1(x))v \geq \alpha(\alpha \lambda + b_1(x))v.$$

The continuous coefficients $b_i$ are bounded on the compact set $\bar{\Omega}$. Therefore there exists $\alpha > 0$ with $Lv > 0$. By linearity of $L$ we obtain $L(u + \epsilon v) > 0$ on $\Omega$ for all $\epsilon > 0$. The continuous functions $u + \epsilon v$ attains on $\bar{\Omega}$ a maximum. The first derivative of the function $u + \epsilon v$ which is twice differentiable on $\Omega$ vanishes at a maximum $x_0 \in \Omega$ and the Hessian is negative semi-definite. In particular there exists an orthogonal matrix $B \in O(n)$ and non-positive $\lambda_1, \ldots, \lambda_n$ with $\frac{\partial^2 (u+\epsilon v)(x_0)}{\partial x_i \partial x_j} = \sum_k B_{ki} \lambda_k B_{kj}$. Now the ellipticity implies $-L(u + \epsilon v)(x_0) \geq -\lambda \sum_{ki} \lambda_k B_{ki}^2 \geq 0$ and contradicts to $L(u + \epsilon v) > 0$. Therefore for all $\epsilon > 0$ the maximum belongs to the boundary $x_0 \in \partial\Omega$:

$$\sup_{x \in \Omega} u(x) + \epsilon \inf_{x \in \Omega} v(x) \leq \sup_{x \in \Omega}(u(x) + \epsilon v(x)) = \max_{x \in \partial\Omega}(u(x) + \epsilon v(x)) \leq \max_{x \in \partial\Omega} u(x) + \epsilon \max_{x \in \partial\Omega} v(x).$$

Because this holds for all $\epsilon > 0$ the boundedness of $v$ on $\bar{\Omega}$ implies the theorem. **q.e.d.**

The negative of the functions $u$ in the theorem obey $Lu \leq 0$ and take a minimum on the boundary. In particular, the solutions $u$ of $Lu = 0$ take the maximum and the minimum on the boundary.

## 3.4   Green's Function

In this section we try to find some conditions which ensure the existence and uniqueness of a harmonic function on a path–connected, open and bounded domain $\Omega \subset \mathbb{R}^n$. A natural candidate for further conditions are boundary value problems. This means that we assume that either the harmonic function or some of its derivatives extends continuously to the boundary and coincides there with a given function. We call a function $u$ on the closure $\bar{\Omega}$ of an domain $m$ times continuously differentiable, if it is $m$ times continuously differentiable on $\Omega$ and all partial derivatives of order at most $m$ extend continuously to $\partial\Omega$.

In the following formula we apply the Divergence Theorem to $x \mapsto v(x)\nabla u(x)$:

**Green's First Formula 3.14.** *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_\Omega v(y)\triangle u(y)d^n y + \int_\Omega \nabla v(y) \cdot \nabla u(y)d^n y = \int_{\partial\Omega} v(z)\nabla u(z) \cdot N d\sigma(z). \qquad \mathbf{q.e.d.}$$

If we subtract the formula for interchanged $u$ and $v$, then we obtain:

**Green's Second Formula 3.15.** *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_\Omega \left(v(y)\triangle u(y) - u(y)\triangle v(y)\right) d^n y = \int_{\partial\Omega} \left(v(z)\nabla u(z) - u(z)\nabla v(z)\right) \cdot N d\sigma(z). \quad \mathbf{q.e.d.}$$

Let us apply this formula to the fundamental solution $v(y) = \Phi(x-y)$. This solution is harmonic only for $y \neq x$. Like in the proof of Theorem 3.2 we restrict the integral to the compliment of $B(x, \epsilon)$. For $u \in C^2(\mathbb{R}^n)$ we showed in the proof of Theorem 3.2

$$\lim_{\epsilon \to 0} \int_{\partial(\mathbb{R}^n \setminus B(x,\epsilon))} u(z)\nabla_z \Phi(x-z) \cdot N d\sigma(z) = \lim_{\epsilon \to 0} \int_{\partial(\mathbb{R}^n \setminus B(0,\epsilon))} u(x-z)\nabla\Phi(z) \cdot N d\sigma(z) = u(x).$$

We also showed that the other integral over $\partial(\mathbb{R}^2 \setminus B(x, \epsilon))$ converges to zero in the limit $\epsilon \to 0$ and $\triangle_y \Phi(x-y)$ vanishes on $\mathbb{R}^n \setminus B(x, \epsilon)$. This proves

**Green's Representation Theorem 3.16.** *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for $x \in \Omega$ and a function $u \in C^2(\bar{\Omega})$ we have*

$$u(x) = -\int_\Omega \Phi(x-y)\triangle u(y)d^n y + \int_{\partial\Omega} \left(\Phi(x-z)\nabla u(z) - u(z)\nabla_z \Phi(x-z)\right) \cdot N d\sigma(z).$$

This implies that on $\Omega$ each solution of the Poisson equation is uniquely determined by the values of $u$ and the normal derivative $\nabla u \cdot N$ on $\partial\Omega$. Conversely, we look for such functions, such that there exists a solution of Poisson's equation with the additional condition that $u$ and $\nabla u \cdot N$ take on $\partial\Omega$ the given values. The Weak Maximum Principle implies the harmonic function is already uniquely determined by the values of $u$ on $\partial\Omega$. So we formulate the following boundary value problem:

**Dirichlet Problem 3.17.** *For a given function $f$ on an open domain $\Omega \subset \mathbb{R}^n$ and $g$ on $\partial\Omega$ we look for a solution $u$ of $-\triangle u = f$ on $\Omega$ which extends continuously to $\partial\Omega$ and coincides there with $g$.*

**Green's Function 3.18.** *A function $G_\Omega : \{(x, y) \in \Omega \times \Omega \mid x \neq y\} \to \mathbb{R}$ is called Green's function for the open domain $\Omega \subset \mathbb{R}^n$, if it has the following two properties:*

**(i)** *For $x \in \Omega$ the function $y \mapsto G_\Omega(x, y) - \Phi(x - y)$ is harmonic on $y \in \Omega$.*

**(ii)** *For $x \in \Omega$ the function $y \mapsto G_\Omega(x, y)$ extends continuously to $\partial\Omega$ and vanishes on $y \in \partial\Omega$.*

Green's Second Formula yields for the function $v(y) = G_\Omega(x, y) - \Phi(x - y)$:

$$-\int_\Omega \Phi(x - y)\triangle u(y)d^n y + \int_{\partial\Omega} (\Phi(x - z)\nabla u(z) - u(z)\nabla_z\Phi(x - z)) \cdot N d\sigma(z)$$

$$= -\int_\Omega G_\Omega(x, y)\triangle u(y)d^n y - \int_{\partial\Omega} u(z)\nabla_z G_\Omega(x, z) \cdot N d\sigma(z).$$

Now Green's Representation Theorem implies

$$u(x) = -\int_\Omega G_\Omega(x, y)\triangle_y u(y)d^n y - \int_{\partial\Omega} u(z)\nabla_z G_\Omega(x, z) \cdot N d\sigma(z).$$

If, conversely, the functions $f : \bar\Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ have sufficient regularity, then

$$u(x) = \int_\Omega G_\Omega(x, y)f(y)d^n y - \int_{\partial\Omega} g(z)\nabla_z G_\Omega(x, z) \cdot N d\sigma(z)$$

solves the Dirichlet Problem. In fact by Theorem 3.2 the first term solves the Dirichlet Problem for $g = 0$. If $g : \partial\Omega \to \mathbb{R}$ is the boundary value of a function on $\Omega$ with sufficient regularity, then the difference of $g$ minus the corresponding first term is harmonic and coincides with the second term. Therefore the Dirichlet Problem reduces to the search of the Green's Function.

For $x \in \Omega$ the difference $y \mapsto G_\Omega(x, y) - \Phi(x - y)$ is harmonic on $y \in \Omega$ and on the boundary $\partial\Omega$ equal to $-\Phi(x - y)$. Hence the difference is the solution of the Dirichlet Problem for $f = 0$ and $g(x) = -\Phi(x - y)$.

**Theorem 3.19** (Symmetry of the Green's Function). *If there is a Green's Function $G_\Omega$ for the domain $\Omega$, then $G_\Omega(x, y) = G_\Omega(y, x)$ holds for all $x \neq y \in \Omega$.*

*Proof.* For $x \neq y \in \Omega$ let $\epsilon > 0$ be sufficiently small, such that both balls $B(x, \epsilon)$ and $B(y, \epsilon)$ are disjoint subsets of $\Omega$. Green's Second Formula implies for the domain $\Omega \setminus (B(x, \epsilon) \cup B(y, \epsilon))$ and the functions $u(z) = G_\Omega(x, z)$ and $v(z) = G_\Omega(y, z)$

$$\int_{\partial B(x,\epsilon)} (G_\Omega(y, z)\nabla_z G_\Omega(x, z) - G_\Omega(x, z)\nabla_z G_\Omega(y, z)) \cdot N d\sigma(z)$$

$$= \int_{\partial B(y,\epsilon)} (G_\Omega(x, z)\nabla_z G_\Omega(y, z) - G_\Omega(y, z)\nabla_z G_\Omega(x, z)) \cdot N d\sigma(z).$$

For $\epsilon \to 0$ the estimate for $L_\epsilon$ in the proof of Theorem 3.2 shows that both second terms converge to zero. The calculation of $K_\epsilon$ in the proof of Theorem 3.2 carries over and shows that the first terms converge to $G_\Omega(y, x)$ and $G_\Omega(x, y)$, respectively. **q.e.d.**

We shall calculate Green's function for all balls in $\mathbb{R}^n$. Let us first restrict to the unit ball $\Omega = B(0,1)$. We may use the inversion $x \mapsto \tilde{x} = \frac{x}{|x|^2}$ in the unit sphere $\partial B(0,1)$ in order to solve the corresponding Dirichlet Problem. The inversion maps the inside of the unit ball to the outside and vice versa. It helps to solve the Dirichlet Problem for $f = 0$ and $g(x) = \Phi(x-y)$:

**Green's Function of the unit ball 3.20.** *The Green's Function of $B(0,1)$ is*

$$G_{B(0,1)}(x,y) = \Phi(x-y) - \Phi(|x|(\tilde{x}-y)) = \begin{cases} \Phi(x-y) - |x|^{2-n}\Phi(\tilde{x}-y) & \text{for } n > 2 \\ \Phi(x-y) - \Phi(\tilde{x}-y) - \Phi(x) & \text{for } n = 2. \end{cases}$$

*Proof.* For $|y| = 1$ we have $|x|^2|\tilde{x}-y|^2 = 1 - 2y \cdot x + |x|^2 = |x-y|^2$. Hence $\Phi(|x|(\tilde{x}-y))$ and $\Phi(x-y)$ coincide on the boundary $y \in \partial B(0,1)$. **q.e.d.**

**Poisson's Representation Formula 3.21.** *For $f \in C^2(\overline{B(z,r)})$ and $g \in C(\partial B(z,r))$ the unique solution of the Dirichlet Problem on $\Omega = B(z,r)$ is given by*

$$u(x) = \frac{1}{r^{n-2}} \int_{B(z,r)} G_{B(0,1)}\left(\tfrac{x-z}{r}, \tfrac{y-z}{r}\right) f(y) d^n y + \frac{1 - \frac{|x-z|^2}{r^2}}{n\omega_n} \int_{\partial B(0,1)} \frac{g(z+ry)}{|\frac{x-z}{r} - y|^n} d\sigma(y).$$

*Proof.* The affine map $x \mapsto \frac{x-z}{r}$ is a homeomorphism from $B(z,r)$ onto $B(0,1)$ and from $\partial B(z,r)$ onto $\partial B(0,1)$. The difference $r^{2-n}\Phi\left(\frac{x-z}{r} - \frac{y-z}{r}\right) - \Phi(x-y)$ vanishes for $n > 2$ and is constant for $n = 2$. Therefore the Green's function of $B(z,r)$ is equal to

$$G_{B(z,r)}(x,y) = r^{2-n}G_{B(0,1)}\left(\tfrac{x-z}{r}, \tfrac{y-z}{r}\right).$$

It suffices to consider the two cases $g = 0$ and $f = 0$ separately. The properties of the Green's function together with Theorem 3.2 show, that for $g = 0$ the function $u$ differs by a harmonic function from a solution of Poisson's equation. By the symmetry of the Green's Function the map $x \mapsto G_{B(z,r)}(y,x)$ extends continuously to $\overline{B(z,r)}$ and vanishes on the boundary $x \in \partial B(z,r)$. This finishes the proof for $g = 0$.

For $|y| = 1$ and $n > 2$ we observe (the reader should check this formula for $n = 2$):

$$K(x,y) = -\nabla_y G_{B(0,1)}(x,y) \cdot \frac{y}{|y|}$$

$$= \frac{-1}{n(n-2)\omega_n} \frac{y}{|y|} \cdot \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}|\tilde{x}-y|^{n-2}} \right)$$

$$= \frac{1}{n\omega_n} \frac{y}{|y|} \cdot \left( \frac{y-x}{|x-y|^n} - \frac{|x|^2(y-\tilde{x})}{|x|^n|\tilde{x}-y|^n} \right)$$

$$\text{(for } |y| = 1) = \frac{1 - x \cdot y - |x|^2 + x \cdot y}{n\omega_n|x-y|^n} = \frac{1 - |x|^2}{n\omega_n|x-y|^n}.$$

By the Symmetry of the Green's Function the function $x \mapsto K(x, y)$ is harmonic. Hence for $f = 0$ the given function $u$ is harmonic. For finishing the proof we show that

$$u(z + rx) = \int_{\partial B(0,1)} g(z + ry)K(x, y)d\sigma(y)$$

extends continuously to $x \in \partial B(0, 1)$ and coincides there with $g(z + rx)$. We observe

  (i) the integral kernel $K(x, y)$ is positive for $(x, y) \in B(0, 1) \times \partial B(0, 1)$.

  (ii) For all $x \in \partial B(0, 1)$ and $\epsilon > 0$ the family of functions $y \mapsto K(\lambda x, y)$ converge uniformly to zero for $\lambda \uparrow 1$ on $y \in \partial B(0, 1) \setminus B(x, \epsilon)$, and

  (iii) The formula which follows from Green's Second Formula and Green's Representation Formula yields for the function $u = 1$ on the domain $\Omega = B(0, 1)$

$$\int_{\partial B(0,1)} K(x, y)d\sigma(y) = 1 \quad \text{for} \quad x \in B(0, 1).$$

For continuous $g$ the properties (i)-(iii) ensure that in the limit $\lambda \uparrow 1$ the family of functions $x \mapsto \int_{\partial B(0,1)} g(y)K(\lambda x, y)d\sigma(y)$ converge on $\partial B(0, 1)$ uniformly to $g$.  **q.e.d.**

A harmonic function $u$ on $B(z, r)$ which extends continuously to $\partial B(z, r)$ obeys

$$u(x) = \frac{1 - \frac{|x-z|^2}{r^2}}{n\omega_n} \int_{\partial B(0,1)} \frac{u(z + ry)}{|\frac{x-z}{r} - y|^n} d\sigma(y) = \frac{r^2 - |x - z|^2}{nr\omega_n} \int_{\partial B(z,r)} \frac{u(y)}{|x - y|^n} d\sigma(y).$$

In particular, $u$ is completely determined by the values on $\partial B(z, r)$. Partial derivatives with respect to $x$ yield similar formulas for the values of all partial derivatives of $u$. This formula implies the Mean Value property. For all $y \in \partial B(z, r)$ the Taylor series of $x \mapsto |x - y|^{-n} = (y^2 - 2xy + x^2)^{-\frac{n}{2}}$ in $x = z$ converge on all balls $B(z, r')$ with $r' < r$ uniformly to $|x - y|^{-n}$. Consequently all harmonic functions are analytic.

**Corollary 3.22.** *Harmonic functions on an open domain $\Omega \subset \mathbb{R}^n$ are analytic.***q.e.d.**

**Exercise 3.23.**    *1. Show the estimate $|\partial^\alpha |x|^{-n}| \leq C(n, |\alpha|)|x|^{-n-|\alpha|}$ for all $|x| \neq 0$ and all multi-indices $\alpha$ with a constant $C(n, |\alpha|)$ depending only on $n$ and $|\alpha|$.*

  *2. Give another proof of Corollary 3.4.*

**Lemma 3.24.** *Let $\Omega \subset \mathbb{R}^n$ be an open neighbourhood of $0$ and $u$ a bounded harmonic function on $\Omega \setminus \{0\}$. Then $u$ extends as a harmonic function to $\Omega$.*

*Proof.* On a ball $B(0,r)$ with compact closure in $\Omega$, Theorem 3.21 gives a harmonic function $\tilde{u}$ which coincides on $\partial B(0,r)$ with $u$. The family of harmonic functions $u_\epsilon(x) = \tilde{u}(x) - u(x) + \epsilon G_{B(0,r)}(x, 0)$ on $B(0,r) \setminus \{0\}$ vanish on $\partial B(0,r)$. If for any $\epsilon > 0$ the function $u_\epsilon$ takes on $B(0,r) \setminus \{0\}$ a negative value, then due to the boundedness of $u$ and $\tilde{u}$ and the unboundedness of $G_{B(0,r)}(\cdot, 0)$ the harmonic function $u_\epsilon$ has a negative minimum on $B(0,r) \setminus \{0\}$. This contradicts the Strong Maximum Principle. Hence $u_\epsilon$ is non-negative. Analogously $u_\epsilon$ us for negative $\epsilon$ non-positive. Otherwise $u_\epsilon$ would have a positive maximum in $B(0,r) \setminus \{0\}$. In both limits $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$ $u_0 = \tilde{u} - u$ vanishes identically on $B(0,r) \setminus \{0\}$ and $\tilde{u}$ is a harmonic extension of $u$ to $\Omega$. **q.e.d.**

The proof shows a slightly stronger statement. Each harmonic function on $\Omega \setminus \{0\}$ whose absolute value $|u(x)|$ is for all $\epsilon > 0$ bounded by $\epsilon G_{B(0,r)}(x, 0)$ on $B(0,\delta) \setminus \{0\}$ with sufficiently small $\delta > 0$ depending on $\epsilon$ has an harmonic extension to $\Omega$.

## 3.5 Dirichlet's Principle

The unique solution of Dirichlet's Problem solves also a variational problem.

**Dirichlet's Prinzip 3.25.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and obey the assumptions of the Divergence Theorem. For continuous real functions $f$ on $\bar{\Omega}$ and $g$ on $\partial\Omega$ the solution $u$ of the Dirichlet Problem 3.4 is the minimizer of the following functional:*

$$I : \{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\} \to \mathbb{R}, \quad w \mapsto I(w) = \int_\Omega \left( \frac{1}{2} \nabla w \cdot \nabla w - wf \right) d^n x.$$

*Proof.* Let $u$ be a solution of the Dirichlet Problem and $w$ another function in the domain $\{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\}$ of $I$. An integration by parts yields

$$0 = \int_\Omega (-\triangle u - f)(u - w) d^n x = \int_\Omega \left( \nabla u \cdot \nabla(u - w) - f(u - w) \right) d^n x.$$

$$\int_\Omega (\nabla u \cdot \nabla u - fu) \, d^n x \;=\; \int_\Omega (\nabla u \cdot \nabla w - fw) \, d^n x \leq$$

$$\leq \; \int_\Omega \frac{1}{2} \nabla u \cdot \nabla u d^n x + \int_\Omega \left( \frac{1}{2} \nabla w \cdot \nabla w - fw \right) d^n x$$

Here we used the Cauchy-Schwarz inequality:

$$\int_\Omega \nabla u \cdot \nabla w d^n x \leq \int_\Omega \nabla u \cdot \nabla w d^n x + \frac{1}{2} \int_\Omega (\nabla u - \nabla w) \cdot (\nabla u - \nabla w) d^n x =$$

$$\int_\Omega \frac{1}{2} \nabla u \cdot \nabla u d^n x + \int_\Omega \frac{1}{2} \nabla w \cdot \nabla w d^n x.$$

This shows $I(u) \leq I(w)$.

If, conversely, $u$ is a minimum, then all $v \in C^2(\bar{\Omega})$ which vanish on $\partial\Omega$ obey

$$0 = \frac{d}{dt}I(u+tv)\Big|_{t=0} = \frac{d}{dt}\left(I(u) + t\int_\Omega (\nabla u \cdot \nabla v - fv)\, d^n x + \frac{t^2}{2}\int_\Omega \nabla v \cdot \nabla v d^n x\right)\Big|_{t=0}$$

$$= \int_\Omega (\nabla u \cdot \nabla v - fv)\, d^n x = \int_\Omega (-\triangle u - f)v d^n x.$$

The final integration by parts shows $-\triangle u = f$ on $\Omega$.                           **q.e.d.**

Finally we remark that one can also prove the uniqueness of the solution with the help of this functional. The difference of two solutions solves the Dirichlet Problem for $f = 0$ and $g = 0$. In this case the functional is non-negative, and vanishes if and only if $u$ is constant. The boundary conditions forces this constant to be zero. By using this principle one can show in a larger class of functions, that this functional has a unique minimizer, which thereby solves the Dirichlet Problem.