Chapter 1

First Order PDEs

In this introductory chapter we first introduce partial differential equations and then consider first order partial differential equations. We shall see that they are simpler than higher order partial differential equations. In contrast to higher order partial differential equations these first order partial differential equations are similar to ordinary differential equations and can be solved by using the theory of ordinary differential equations. After this introductory chapter we shall focus on second order partial differential equations. Before we consider the three main examples of second order differential equations we introduce some general concepts in the next chapter. These general concepts are partially motivated by observations contained in the first chapter.

A partial differential equation is an equation on the partial derivatives of a function depending on at least two variables.

Definition 1.1. *A possibly vector valued equation of the following form*

$$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0$$

is called partial differential equation of order k. Here F is a given function and u an unknown function. The expressions $D^k u$ denotes the vector of all partial derivatives of the function u of order k. The function u is called a solution of the differential equation, if u is k times differentiable and obeys the partial differential equation.

On open subsets $\Omega \subset \mathbb{R}^n$ we denote the partial derivatives of higher order by $\partial^\gamma = \prod_i \partial_i^{\gamma_i} = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}$ with multiindices $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$. The multiindices are ordered by $\delta \leq \gamma \iff \delta_i \leq \gamma_i$ for $i = 1, \ldots, n$. The partial derivative acts only on the immediately following function; they only act on a product of functions if the product is grouped together in brackets.

Exercise 1.2. *Show for all $\gamma \in \mathbb{N}_0^n$ the generalised Leibniz rule*

$$\partial^\gamma(uv) = \sum_{0\le\delta\le\gamma} \binom{\gamma}{\delta} \partial^\delta u \partial^{\gamma-\delta} v := \sum_{\delta_1=0}^{\gamma_1} \binom{\gamma_1}{\delta_1} \dots \sum_{\delta_n=0}^{\gamma_n} \binom{\gamma_n}{\delta_n} \partial^\delta u \partial^{\gamma-\delta} v.$$

1.1 Homogeneous Transport Equation

One of the simplest partial differential equations is the transport equation:

$$\dot{u} + b \cdot \nabla u = 0.$$

Here $\dot{u}$ denotes the partial derivative $\frac{\partial u}{\partial t}$ of the unknown function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $b \in \mathbb{R}^n$ is a vector, and the product $b \cdot \nabla u$ denotes the scalar product of the vector b with the vector of the first partial derivatives of u with respect to x:

$$b \cdot \nabla u(x,t) = b_1 \frac{\partial u(x,t)}{\partial x_1} + \ldots + b_n \frac{\partial u(x,t)}{\partial x_n}.$$

Let us first assume that $u(x,t)$ is a differentiable solution of the transport equation. For all fixed $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ the function

$$z(s) = u(x + s \cdot b, t + s)$$

is a differentiable function on $s \in \mathbb{R}$, whose first derivative vanishes:

$$z'(s) = b\nabla u(x + s \cdot b, t + s) + \dot{u}(x + s \cdot b, t + s) = 0.$$

Therefore u is constant along all parallel straight lines in direction of $(b, 1)$. Furthermore, u is completely determined by the values on all these parallel straight lines.

Initial Value Problem 1.3. *We are looking for a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the transport equation with given b, which at $t = 0$ is equal to some given function $g : \mathbb{R}^n \to \mathbb{R}$.*

All parallel straight lines in direction of $(b, 1)$ intersect $\mathbb{R}^n \times \{0\}$ exactly once:

$$(x + sb, t + s) \in \mathbb{R}^n \times \{0\} \iff s = -t$$

Hence the solution has to be equal to $u(x,t) = g(x - tb)$. If g is differentiable on $\mathbb{R}^n$, then this function indeed solves the transport equation. In this case the initial value problem has a unique solution. Otherwise, if g is not differentiable on $\mathbb{R}^n$, then the initial value problem does not have a solution. As we have seen above, whenever the initial value problem has a solution, then the function $u(x,t) = g(x - bt)$ is the unique solution. So it might be that this candidate is a solution in a more general sense. In fact in the next chapter we shall see in Exercise 2.10 that in case of generalised differentiable functions g which include all continuous functions, the function $u(x,t) = g(x - bt)$ is the unique solution.

1.2 Inhomogeneous Transport Equation

Now we consider the corresponding inhomogeneous transport equation:

$$\dot{u} + b \cdot \nabla u = f.$$

Again $b \in \mathbb{R}^n$ is a given vector, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a given function and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the unknown function.

Initial Value Problem 1.4. *Given a vector $b \in \mathbb{R}$, a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and an initial value $g : \mathbb{R}^n \to \mathbb{R}$, we are looking for a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the inhomogeneous transport equation which is at $t = 0$ equal to g.*

We define for each $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ the function $z(s) = u(x + sb, t + s)$ which solves

$$z'(s) = b \cdot \nabla u(x + sb, t + s) + \dot{u}(x + sb, t + s) = f(x + sb, t + s).$$

This function obeys

$$u(x,t) - g(x - bt) = z(0) - z(-t) = \int_{-t}^{0} z'(s)ds = \int_{-t}^{0} f(x + sb, t + s)ds = \int_{0}^{t} f(x + (s - t)b, s)ds.$$

Hence the solution u is equal to $\quad u(x,t) = g(x - bt) + \int_{0}^{t} f(x + (s - t)b, s)ds.$

We observe that this formula is analogous to the formula for solutions of inhomogeneous initial value problems of linear ODEs. The unique solution is the sum of the unique solution of the corresponding homogeneous initial value problem and the integral over solution of the homogeneous equation with the inhomogeneity as initial values. Again one can show that the initial value problem has a unique solution in a generalised sense if the initial value is a generalised differentiable function. We obtained these solutions of the first order homogeneous and inhomogeneous transport equation by solving an ODE. We shall generalise this method in Section 1.5 and solve more general first order PDEs by solving an appropriate chosen system of first order ODEs.

1.3 Scalar Conservation Laws

In this section we consider the non-linear first order differential equation

$$\dot{u}(x,t) + \frac{\partial f(u(x,t))}{\partial x} = \dot{u}(x,t) + f'(u(x,t)) \cdot \frac{\partial u(x,t)}{\partial x} = 0$$

with a smooth function $f : \mathbb{R} \to \mathbb{R}$. Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the unknown function. This equation is a non-linear first order PDE, and the method of characteristic applies. We impose the initial condition $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$ with some given function $u_0 : \mathbb{R} \to \mathbb{R}$. For any compact interval $[a,b]$ we calculate

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \dot{u}(x,t)dx = -\int_a^b \frac{\partial f(u(x,t))}{\partial x}dx = f(u(a,t)) - f(u(b,t)).$$

This is the meaning of a conservation law: the change of the integral of $u(\cdot,t)$ over $[a,b]$ is equal to the 'flux' of $f(u(x,t))$ through the 'boundary' $\partial[a,b] = \{a,b\}$.

For any $x \in \mathbb{R}$ we solve the ordinary differential equation $x'(s) = f'(u(x(s),s))$ with initial value $x(0) = x$. Consequently the derivative of the function $z(s) = u(x(s),s)$ is

$$z'(s) = \frac{\partial u(x(s),s)}{\partial x}x'(s) + \dot{u}(x(s),s) = \frac{\partial u(x(s),s)}{\partial x}f'(u(x(s),s)) + \dot{u}(x(s),s) = 0.$$

Hence z is constant and equal to $z(s) = u(x(0),0) = u(x,0) = u_0(x)$ and therefore the derivative of $x(s)$ is constant equal to $x'(s) = f'(u(x,0)) = f'(u_0(x))$. The unique solution of the corresponding initial value problem is $x(s) = x + sf'(u_0(x))$. This implies

$$u(x + tf'(u_0(x)), t) = u_0(x) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2.$$

The solutions for initial values $x_1, x_2 \in \mathbb{R}^n$ with $u_0(x_1) \neq u_0(x_2)$ might intersect at $t \in \mathbb{R}^+$. In this case the method of characteristic implies $u_0(x_1) = u(x_1 + tf'(u_0(x_1)), t) = u(x_2 + tf'(u_0(x_2)), t) = u_0(x_2)$, which is impossible. This intersection of solutions of the characteristic equations is called crossing characteristics. There is a crossing of characteristics for $f'(u_0(x_2)) < f'(u_0(x_1))$ with $x_2 > x_1$.

Theorem 1.5. *If $f \in C^2(\mathbb{R},\mathbb{R})$ and $u_0 \in C^1(\mathbb{R},\mathbb{R})$ with $f''(u_0(x))u_0'(x) > -\alpha$ for all $x \in \mathbb{R}$ and some $\alpha \geq 0$, then there is a unique C^1-solution of the initial value problem*

$$\frac{\partial u(x,t)}{\partial t} + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0 \quad \textit{with} \quad u(x,0) = u_0(x)$$

on $(x,t) \in \mathbb{R} \times [0,\alpha^{-1})$ for $\alpha > 0$ and on $(x,t) \in \mathbb{R} \times [0,\infty)$ for $\alpha = 0$.

Proof. By the method of characteristic the solution $u(x,t)$ is on the lines $x+tf'(u_0(x))$ equal to $u_0(x)$. For all $t \geq 0$ with $1-t\alpha > 0$ the derivative of $x \mapsto x+tf'(u_0(x))$ obeys

$$1 + tf''(u_0(x))u_0'(x) \geq 1 - t\alpha > 0.$$

This implies $\lim_{x\to\pm\infty} x+tf'(u_0(x)) = \pm\infty$. So $x \mapsto x+tf'(u_0(x))$ is C^1-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Therefore there exists for any $y \in \mathbb{R}$ a unique x with $x+tf'(u_0(x)) = y$. Then $u(y,t) = u_0(x)$ solves the initial value problem. **q.e.d.**

Example 1.6. *For $n = 1$ and $f(u) = \frac{1}{2}u^2$ we obtain Burgers equation:*

$$\dot{u}(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0.$$

The solutions of the corresponding characteristic equations are $x(t) = x_0 + u_0(x_0)t$. Therefore the solutions of the corresponding initial value problem obey

$$u(x + tu_0(x), t) = u_0(x).$$

If u_0 is continuously differentiable and monotonic increasing, then for all $t \in [0,\infty)$ the map $x \mapsto x + tu_0(x)$ is a C^1-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there is a unique C^1-solution on $\mathbb{R} \times [0,\infty)$. More generally, if $u_0'(x) > -\alpha$ with $\alpha \geq 0$, then there is a unique C^1-solution on $\mathbb{R} \times [0,\alpha^{-1})$ for $\alpha > 0$ and $(x,t) \in \mathbb{R} \times [0,\infty)$ for $\alpha = 0$.

1.4 Weak Solutions

In this section we look for more general notions of solutions which allow us to extend solutions across the crossing characteristics. For this purpose we use the conserved integrals. Since we will restrict ourselves to the one-dimensional situation for the moment, the natural domains are intervals $\Omega = [a,b]$ with $a < b \in \mathbb{R}$. In this case the conservation law implies

$$\frac{d}{dt}\int\limits_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

Now we look for functions u with discontinuities along the graph $\{(x,t) \mid x = y(t)\}$ of a C^1–function y. In the case that $y(t)$ belongs to $[a,b]$, we split the integral over $[a,b]$ into the integrals over $[a,b] = [a,y(t)] \cup [y(t),b]$. In such a case let us calculate the derivative of the integral over $[a, b]$:

$$\frac{d}{dt}\int_a^b (u(x,t)dx = \frac{d}{dt}\int_a^{y(t)} u(x,t)dx + \frac{d}{dt}\int_{y(t)}^b u(x,t)dx =$$

$$= \dot{y}(t)\lim_{x\uparrow y(t)} u(x,t) + \int_a^{y(t)} \dot{u}(x,t)dx - \dot{y}(t)\lim_{x\downarrow y(t)} u(x,t) + \int_{y(t)}^b \dot{u}(x,t)dx.$$

We abbreviate $\lim_{x\uparrow y(t)} u(x,t)$ as $u^l(y(t),t)$ and $\lim_{x\downarrow y(t)} u(x,t)$ as $u^r(y(t),t)$ and assume that on both sides of the graph of y the function u is a classical solution of the conservation law:

$$\frac{d}{dt}\int_a^b (u(x,t)dx = \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) - \int_a^{y(t)} \frac{d}{dx}f(u(x,t))dx - \int_{y(t)}^b \frac{d}{dx}f(u(x,t))dx$$

$$= \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) + f(u(a,t)) - f(u(b,t)) + f(u^r(y(t),t) - f(u^l(y(t),t).$$

Hence the integrated version of the conservation law still holds, if the following Rankine-Hugonoit condition is fulfilled:

$$\dot{y}(t) = \frac{f(u^r(y,t)) - f(u^l(y,t)}{u^r(y,t) - u^l(y,t)}.$$

Example 1.7. *We consider Burgers equation* $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ *for* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ *with the following continuous initial values* $u(x,0) = u_0(x)$ *and*

$$u_0(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1 - x & \text{for } 0 \le x < 1 \\ 0 & \text{for } 1 \le x. \end{cases}$$

The first crossing of characteristics happens for $t = 1$*:*

$$x + tu_0(x) = \begin{cases} x + t & \text{for } x \le 0, \\ x + t(1 - x) & \text{for } 0 < x < 1, \\ x & \text{for } 1 \le x. \end{cases}$$

For $t < 1$ *the evaluation at* t *is a homeomorphism from* $\mathbb{R}$ *onto itself with inverse*

$$x \mapsto \begin{cases} x - t & \text{for } x \le t, \\ \frac{x-t}{1-t} & \text{for } t < x < 1, \\ x & \text{for } 1 \le x. \end{cases}$$

Therefore the solution is for $0 < t < 1$ *equal to*

$$u(x,t) = \begin{cases} 1 & \text{for } x < t, \\ \frac{x-1}{t-1} & \text{for } t < x < 1, \\ 0 & \text{for } 1 \leq x. \end{cases}$$

At $t = 1$ *the solutions of the characteristic equations starting at* $x \in [0,1]$ *all meet at* $x = 1$*. For* $t > 1$ *there exists a unique discontinuous solution satisfying the Rankine-Hugonoit condition. For small* x *this solution is* 1 *and for large* x *it is* 0*. The corresponding regions has to be separated by a path with velocity* $\frac{1}{2}$ *which starts at* $(x,t) = (1,1)$*. For* $t \geq 1$ *this discontinuous solution is equal to*

$$u(x,t) = \begin{cases} 1 & \text{for } x < 1 + \frac{t-1}{2}, \\ 0 & \text{for } 1 + \frac{t-1}{2} < x. \end{cases}$$

The second initial value problem is not continuous but monotonic increasing. For continuous monotonic increasing functions u_0 the evaluation at t of the solutions of the characteristic equation would be a homeomorphism for all $t > 0$. Therefore in such cases there exists a unique continuous solution for all $t > 0$. But for non-continuous initial values this is not the case.

Example 1.8. *We again consider Burgers equation* $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ *for* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ *with the following non-continuous initial values* $u(x,0) = u_0(x)$ *and*

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 < x. \end{cases}$$

Again there is a unique discontinuous solution which is for small x *equal to* 0 *and for large* x *equal to* 1*. By the Rankine-Hugonoit condition both regions are separated by a path with velocity* $\frac{1}{2}$*. This solution is equal to*

$$u(x,t) = \begin{cases} 0 & \text{for } x < \frac{t}{2}, \\ 1 & \text{for } \frac{1}{2} < x. \end{cases}$$

But there exists another continuous solution, which clearly also satisfies the Rankine-Hugonoit condition:

$$u(x,t) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{t} & \text{for } 0 < x < t, \\ 1 & \text{for } 1 \leq x. \end{cases}$$

These solutions are constant along the lines $x = ct$ for $c \in [0, 1]$. These lines all intersect in the discontinuity at $(x, t) = (0, 0)$. Besides these two extreme cases there exists infinitely many other solutions with several regions of discontinuity, which all satisfy the Rankine-Hugonoit condition.

These examples show that such weak solutions exists for all $t \geq 0$ but are not unique. We now restrict the space of weak solutions such that they have a unique solutions for all $t \geq 0$. Since we want to maximise the regularity we only accept discontinuities if there are no continuous solutions. In the last example we prefer the continuous solution. So for Burgers equation this means we only accept discontinuous solutions, which take larger values for smaller x and smaller values for larger x.

Definition 1.9 (Lax Entropy condition)**.** *A discontinuity of a weak solution along a C^1–path $t \mapsto y(t)$ satisfies the Lax entropy condition, if along the path the following inequality is fulfilled:*

$$f'(u^l(y,t)) > \dot{y}(t) > f'(u^r(y,t)).$$

A weak solutions with discontinuities along C^1–paths is called an admissible solution, if along the path both the Rankine–Hugonoit condition and the Lax Entropy condition are satisfied.

For continuous u_0 there is a crossing of characteristics if $f'(u_0(x_1)) > f'(u_0(x_2))$ for $x_1 < x_2$. So this condition ensures that discontinuities can only show up if we cannot avoid a crossing of characteristics.

Theorem 1.10. *Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be convex and u and v two admissible solutions of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0.$$

in $L^1(\mathbb{R})$. Then $t \mapsto \|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})}$ is monotonically decreasing.

Proof. We divide $\mathbb{R}$ into maximal intervals $I = [a(t), b(t)]$ with the property that either $u(x,t) > v(x,t)$ or $v(x,t) > u(x,t)$ for all $x \in (a(t), b(t))$. This means that either $x \mapsto u - v$ vanishes at the boundary, or is discontinuous and changes sign at the boundary. We claim that the boundaries $a(t)$ and $b(t)$ of these maximal intervals are differentiable. We prove this only for $a(t)$. For $b(t)$ the proof is analogous. To simplify notation we write a and b instead of $a(t)$ and $b(t)$. If either $u(\cdot, t)$ or $v(\cdot, t)$ is discontinuous at a, then by definition of an admissible solution the locus of the discontinuity a is differentiable with respect to t. If u and v are both continuously differentiable at (a, t) with $u(a, t) = v(a, t)$, then by the method of characteristic for sufficiently small $\epsilon > 0$ all $x \in (a - \epsilon, a + \epsilon)$ with $u = v$ preserve this property along characteristic lines $x + tf'(u(x(t),t) = x + tf'(v((x(t),t)$. So along these lines also the properties $u \neq v$ and $u > v$ are preserved. This implies that a is differentiable with $\dot{a}(t) = f(u(a,t)) = f(v(a,t))$. We only consider intervals on whose interior $u > v$. On the other intervals these arguments apply with interchanged u and v. Now we calculate

$$\frac{d}{dt}\int_{a(t)}^{b(t)} (u(x,t) - v(x,t))dx$$
$$= \int_{a(t)}^{b(t)} (\dot{u}(x,t) - \dot{v}(x,t))dx + \dot{b}(u(b,t) - v(b,t)) - \dot{a}(t)(u((a,t) - v(a,t))$$
$$= \int_{a(t)}^{b(t)} \frac{d}{dx}(f(v(x,t) - f(u(x,t))dx + \dot{b}(t)(u(b,t) - v(b,t)) - \dot{a}(t)(u((a,t) - v(a,t))$$
$$= f(v(b,t) - f(u(b,t) + \dot{b}(t)(u(b,t) - v(b,t)) + f(u(a,t) - f(v(a,t) + \dot{a}(t)(v(a,t) - u(a,t)).$$

If u and v are both differentiable at (a,t), then they take the same values at (a,t) and the corresponding terms in the last line vanishes. The same holds, if u and v are both differentiable at (b,t). For convex f the derivative f' is monotonically increasing and the Lax-Entropy condition implies at all discontinuities y of $u(\cdot,t)$ and $v(\cdot,t)$

$$u^l(y,t) > u^r(y,t), \qquad\qquad v^l(y,t) > v^r(y,t),$$

respectively. If one of the two solutions u and v is at the boundary of I continuous and the other is non-continuous, then the value of the continuous solution has to lie in between the limits of the non-continuous solution, because at the boundary either $u - v$ becomes zero or changes sign. For v being continuous and u being discontinuous we would have $u^l(a,t) < v(a,t) < u^r(a,t)$ by $u > v$ on (a,b) in contradiction to the fomrer inequality. So either $u(\cdot,t)$ is continuous and differentiable at a and $v(\cdot,t)$ is discontinuous at $a(t)$ and analogously u is discontinuous at b and v is continuous and differentiable at b. In the first case we use the Rankine Hugonoit condition to determine $\dot{a}(t)$ and $\dot{b}(t)$. The corresponding contribution to the derivative of $\|u(\cdot,t) - v(\cdot,t)\|_1$ is

$$f(u(a,t)) - f(v^r(a,t)) + \dot{a}(t)\left(v^r(a,t) - u(a,t)\right) =$$
$$= f(u(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t)) - f(v^l(a,t))}{v^r(a,t) - v^l(a,t)}\left(v^r(a,t) - u(a,t)\right)$$
$$= f(u(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t) - u(a,t)}{v^l(a,t) - v^r(a,t)} + f(v^l(a,t))\frac{u(a,t) - v^r(a,t)}{v^l(a,t) - v^r(a,t)}\right).$$

Since f is convex the secant lies above the graph of f. Hence due to $u(a,t) \in [v^r(a,t), v^l(a,t)]$ this expression is non-positive. In the second case the contribution to the derivative of $\|u(\cdot,t)-v(\cdot,t)\|_1$ is

$$\begin{aligned}
&f(v(b,t)) - f(u^l(b,t)) + \dot{b}(t)\left(u^l(b,t) - v(b,t)\right) = \\
&\qquad = f(v(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t)) - f(u^l(b,t))}{u^r(b,t) - u^l(b,t)}\left(u^l(b,t) - v(b,t)\right) \\
&\qquad = f(v(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t) - v(b,t)}{u^l(b,t) - u^r(b,t)} + f(u^l(b,t))\frac{v(b,t) - u^r(b,t)}{u^l(b,t) - u^r(b,t)}\right).
\end{aligned}$$

Again due to $v(b,t) \in [u^r(b,t), u^l(b,t)]$ this expression is non-positive.

If finally both solutions are discontinuous at $a(t)$ or $b(t)$. Since $u(\cdot,t) - v(\cdot,t)$ is positive on I, the Lax Entropy condition implies $[u^r(a,t), u^l,(a,t)] \subset [v^r(a,t), v^l(a,t)]$ and $[v^r(b,t), v^l(b,t)] \subset [u^r(b,t), u^l(b,t)]$, respectively. The corresponding contributions to the derivative of $\|u(\cdot,t)-v(\cdot,t)\|_1$ are again non-positive:

$$\begin{aligned}
&f(u^r(a,t)) - f(v^r(a,t)) + \dot{a}(t)\left(v^r(a,t) - u^r(a,t)\right) = \\
&\qquad = f(u^r(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t)) - f(v^l(a,t))}{v^r(a,t) - v^l(a,t)}\left(v^r(a,t) - u^r(a,t)\right) \\
&\qquad = f(u^r(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t) - u^r(a,t)}{v^l(a,t) - v^r(a,t)} + f(v^l(a,t))\frac{u^r(a,t) - v^r(a,t)}{v^l(a,t) - v^r(a,t)}\right).
\end{aligned}$$

$$\begin{aligned}
&f(v^l(b,t)) - f(u^l(b,t)) + \dot{b}(t)\left(u^l(b,t) - v^l(b,t)\right) = \\
&\qquad = f(v^l(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t)) - f(u^l(b,t))}{u^r(b,t) - u^l(b,t)}\left(u^l(b,t) - v^l(b,t)\right) \\
&\qquad = f(v^l(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t) - v^l(b,t)}{u^l(b,t) - u^r(b,t)} + f(u^l(b,t))\frac{v^l(b,t) - u^r(b,t)}{u^l(b,t) - u^r(b,t)}\right).
\end{aligned}$$

Hence the contributions to $\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_1$ of all intervals are non-positive. **q.e.d.**

This implies that admissible solutions are unique, if they exist. By utilising an explicit formula for admissible solutions one can also prove the existence of admissible solutions. The following theorem is Theorem 10.3 in the lecture notes "Hyperbolic Partial Differential Equations" by Peter Lax, Courant Lecture Notes in Mathematics **14**, American Mathematical Society (2006), which also supplies a proof.

Theorem 1.11. *For $f \in C^2(\mathbb{R},\mathbb{R})$ is strictly convex and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there exists an unique admissible solution $u(x,t)$ of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0 \quad \text{and} \quad u(x,0) = u_0(x) \quad \text{for all } x \in \mathbb{R}.$$

1.5 Method of Characteristics

In this section we shall solve the following first order PDE:

$$F(\nabla u(x), u(x), x) = 0.$$

Here u is a real unknown function on an open domain $\Omega \subset \mathbb{R}^n$ and F is a real function on an open subset of $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We try to obtain the solution by solving an appropriate system of first order ODEs for the values of the function u along some integral curves along some vector fields. So let $x(s)$ be such such an integral curve and $p(s) = \nabla u(x(s))$ the gradient of the unknown function along this curve. We want to determine the curve $s \mapsto x(s)$ in such a way, that the triple $s \mapsto (p(s), z(s), x(s))$ with $z(s) = u(x(s))$ solves an ODE. For this purpose we differentiate

$$\frac{dp_i(s)}{ds} = \frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{dx_j(s)}{ds}.$$

The total derivative of $F(\nabla u(x), u(x), x) = 0$ with respect to x_i gives

$$0 = \frac{dF(\nabla u(x), u(x), x)}{dx_i} =$$
$$= \sum_{j=1}^{n} \frac{\partial F(\nabla u(x), u(x), x)}{\partial p_j}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial z}\frac{\partial u(x)}{\partial x_i} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial x_i}.$$

Due to the commutativity $\partial_i \partial_j u = \partial_j \partial_i u$ of the second partial derivatives we obtain

$$\sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}\frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} + \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) + \frac{\partial F(p(s), z(s), x(s))}{\partial x_i} = 0.$$

Now we choose the vector field for the integral curves $s \mapsto x(s)$ as

$$\frac{dx_j}{ds} = \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

This choice allows us to rewrite the differential equation

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{dx_j}{ds}(s)$$

as

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} =$$
$$= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s).$$

Finally we differentiate

$$\frac{dz(s)}{ds} = \frac{du(x(s))}{ds} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x(s)) \frac{dx_j(s)}{ds} = \sum_{j=1}^{h} p_j(s) \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

In this way we indeed obtain the following system of first order ODEs:

$$x_i'(s) = \frac{\partial F(p(s), z(s), x(s))}{\partial p_i}$$
$$p_i'(s) = -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s)$$
$$z'(s) = \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).$$

This is a system of first order ODEs with $2n + 1$ unknown real functions. Let us summarise these calculations in the following theorem:

Theorem 1.12. *Let F be a real differentiable function on an open subset $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a twice differentiable solution on an open subset $\Omega \subset \mathbb{R}^n$ of the first order PDE $F(\nabla u(x), u(x), x) = 0$. For every solution $s \mapsto x(s)$ of the ODE*

$$x_i'(s) = \frac{\partial F}{\partial p_i}(\nabla u(x(s)), u(x(s)), x(s))$$

the functions $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$ solve the ODEs

$$p_i'(s) = -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \text{ and}$$
$$z'(s) = \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).$$

q.e.d.

Now we want to introduce a boundary value problem of the following form:

$$u(y) = g(y) \text{ for all } y \in \Omega \cap H \text{ with } H = \{y \in \mathbb{R}^n \mid y \cdot e_n = x_0 \cdot e_n\}.$$

Here $e_n = (0, \ldots, 0, 1)$ denotes the n–th element of the canonical basis and H the unique hyperplane through $x_0 \in \Omega$ orthogonal to e_n. By the implicit function theorem general continuously differentiable hyper surfaces may be brought into this form by a continuously differentiable coordinate transformation with continuously differentiable inverse: Let $\Phi : \Omega \to \Omega'$ be a continuously differentiable homeomorphism with continuously differentiable inverse Φ^{-1}. Then by the chain rule the composition $u = v \circ \Phi$ of a function $v : \Omega' \to \mathbb{R}$ with Φ obeys for $y = \Phi(x)$ i.e. $x = \Phi^{-1}(y)$

$$\nabla u(x) = \nabla v(\Phi(x)) \cdot \Phi'(x) = \nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right).$$

Here ∇v and ∇u are row vectors and $\Phi'(x)$ the Jacobi matrix. Hence u solves the PDE

$$F(\nabla u(x), u(x), x) = 0$$

if and only if v solves the PDE

$$F\left(\nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right), v(y), \Phi^{-1}(y)\right) = 0.$$

Therefore the PDE for the function v is the zero set of the function

$$G(\nabla v(y), v(y), y) = F\left(\nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right), v(y), \Phi^{-1}(y)\right)$$

In the sequel we assume that the hyperplane H has the following form:

$$H = \{y \in \mathbb{R}^n \mid y \cdot e_n = x_0 \cdot e_n\}.$$

If the hyper surface $H' \subset \Omega'$ is the zero set of a continuously differentiable function $\Lambda : \Omega' \to \mathbb{R}$ whose gradient $\nabla\Lambda$ does not vanish on H', then the implicit function theorem shows that in a neighbourhood of $y_0 \in H'$ there exists such a Φ. Furthermore, Φ is as often differentiable as Λ. In the foregoing theorem the functions u and v has to be twice differentiable. We assume that Φ and Φ^{-1} are twice differentiable. Consequently Λ should be twice differentiable. On $\Omega \cap H$ there must hold

$$F(\nabla u(y), u(y), y) = 0.$$

On order to define initial conditions at $y \in \Omega \cap H$

$$z(0) = g(y), \qquad p(0) = q(y) \quad \text{and} \qquad x(0) = y$$

we have to find a solution $q : \Omega \cap H \to \mathbb{R}^n, \quad y \mapsto q(y)$ of the following equation:

$$F(q(y), g(y), y) = 0 \quad \text{and} \qquad \frac{\partial g(y)}{\partial y_i} = q_i(y) \text{ for } i = 1, \ldots, n-1.$$

The second equations uniquely determine $q_1(y), \ldots, q_{n-1}(y)$ as

$$q_1(y) = \tfrac{\partial g(y)}{\partial y_1}, \ldots, q_{n-1}(y) = \tfrac{\partial g(y)}{\partial y_{n-1}}.$$

It remains to determine the component $q_n(y)$ in such a way, that

$$F(q(y), g(y), y) = 0$$

holds for all $y \in \Omega \cap H$. Now the implicit function theorem implies that this equation implicitly defines a continuously differentiable function $y \mapsto q_n(y)$, if

$$\frac{\partial F(p_0, z_0, x_0)}{\partial p_n} \neq 0.$$

This proves the following lemma:

Lemma 1.13. *Let $F : W \to \mathbb{R}$ and $g : H \to \mathbb{R}$ be continuously differentiable, $x_0 \in \Omega \cap H$, $z_0 = g(x_0)$ and $p_{0,1} = \frac{\partial g(x_0)}{y_1}, \ldots, p_{0,n-1} = \frac{\partial g(x_0)}{y_n}$. If there exists $p_{0,n}$ with*

$$(p_0, z_0, x_0) \in W, \qquad F(p_0, z_0, x_0) = 0 \quad \textit{and} \quad \tfrac{\partial F(p_0,z_0,x_0)}{\partial p_n} \neq 0,$$

then on an open neighbourhood of $x_0 \in \Omega \cap H$ there exists a unique solution q of

$$F(q(y), g(y), y) = 0, \quad q_i(y) = \tfrac{\partial g(y)}{\partial y_i} \textit{ for } i = 1, \ldots, n-1 \quad \textit{and} \quad q(y_0) = p_0. \quad \textbf{q.e.d.}$$

Theorem 1.14. *Let $F : W \to \mathbb{R}$ and $g : \Omega \cap H \to \mathbb{R}$ be three times differentiable functions on open subsets. Furthermore, let $(p_0, z_0, x_0) \in W$ and g satisfy*

$$F(p_0, z_0, x_0) = 0, \quad g(x_0) = z_0, \quad p_{0,1} = \tfrac{\partial g(x_0)}{y_1}, \ldots, \quad p_{0,n-1} = \tfrac{\partial g(x_0)}{y_n}, \quad \tfrac{\partial F}{\partial p_n}(p_0, z_0, x_0) \neq 0.$$

Then there exists on a neighbourhood Ω of x_0 a solution of the boundary value problem

$$F(\nabla u(x), u(x), x) = 0 \quad \textit{for} \quad x \in \Omega \quad \textit{and} \quad u(y) = g(y) \quad \textit{for} \quad y \in \Omega \cap H.$$

Proof. By the foregoing Lemma there exists a solution q on an open neighbourhood of x_0 in H of the following equations

$$F(q(y), g(y), y) = 0, \qquad q_i(y) = \tfrac{\partial g(y)}{\partial y_i} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \quad q(y_0) = p_0.$$

If F is twice and g are three times differentiable then the implicit function theorem yields a twice differentiable solution. The theorem of Picard Lidenlöff shows that the following initial value problem has for all y in the intersection of an open neighbourhood of x_0 with H a unique solution:

$$\begin{aligned} x_i'(s) &= \frac{\partial F}{\partial p_i}(p(s), z(s), x(s)) && \text{with} \quad x(0) = y \\ p_i'(s) &= -\frac{\partial F}{\partial x_i}(p(s), z(s), x(s)) - \frac{\partial F}{\partial z}(p(s), z(s), x(s))p_i(s) && \text{with} \quad p(0) = q(y) \\ z'(s) &= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(p(s), z(s), x(s))p_j(s) && \text{with} \quad z(0) = g(y). \end{aligned}$$

We denote the family of solutions by $(x(y,s), p(y,s), z(y,s))$. For small $\Omega \ni x_0$ there exists an $\epsilon > 0$ such that these solutions are uniquely defined on $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$. Since F and g are three times differentiable all coefficients and initial values are twice differentiable. The theorem on the dependence of solutions of ODEs on the initial values gives that $(y,s) \mapsto (x(y,s), p(y,s), z(y,s))$ is on $(\Omega \cap H) \times (-\epsilon, \epsilon)$ twice differentiable. Due to the choice of the initial values at $s = 0$, the function

$$(\Omega \cap H) \times (-\epsilon, \epsilon) \to \Omega, \quad (y,s) \mapsto x(y,s)$$

has at $(y,s) = (x_0, 0)$ the Jacobi matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_1} \\ & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_{n-1}} \\ 0 & 0 & \dots & 0 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_n} \end{pmatrix}.$$

Since $\frac{\partial F(p_0,z_0,x_0)}{\partial p_n} \neq 0$ this matrix is invertible, and the inverse function theorem implies that on an possibly diminished neighbourhood Ω of x_0 and an appropriately chosen $\epsilon > 0$ this map is a twice differentiable homeomorphism with twice differentiable inverse mapping. Now we define the function $u : \Omega \to \mathbb{R}$ by

$$u(x(y,s)) = z(y,s) \text{ for all } (y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Our next task is to show that this function solves the PDE $F(\nabla u(x), u(x), x) = 0$.

In a first step we observe that the ODE implies

$$\frac{\partial}{\partial s} F(p(y,s), z(y,s), x(y,s)) = 0.$$

Since $F(q(y), g(y), y)$ vanishes for all $y \in \Omega \cap H$ we conclude

$$F(p(y,s), z(y,s), x(y,s)) = 0 \text{ for all } (y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Hence it suffices to show $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$.

In a second step we show

$$\frac{\partial z(y,s)}{\partial s} = \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial s} \quad \text{and} \quad \frac{\partial z(y,s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i}$$

for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$ and all $i = 1, \ldots, n-1$. The first equation follows from the ODE for $x(y,s)$ and $z(y,s)$. For $s = 0$ the second equation follows from the initial conditions for $z(y,s)$, $p(y,s)$ and $x(y,s)$. The derivative of the first equation with respect to y_i yields

$$\frac{\partial^2 z(y,s)}{\partial y_i \partial s} = \sum_{j=1}^{n} \left(\frac{\partial p_j(y,s)}{\partial y_i} \frac{\partial x_j(y,s)}{\partial s} + p_j(y,s) \frac{\partial^2 x_j(y,s)}{\partial y_i \partial s} \right).$$

By the commutativity of the second partial derivatives we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s} \left(\frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&\quad = \frac{\partial^2 z(y,s)}{\partial s \partial y_i} - \sum_{j=1}^{n} \frac{\partial p_j(y,s)}{\partial s} \frac{\partial x_j(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s) \frac{\partial^2 x_j(y,s)}{\partial s \partial y_i} \\
&\quad = \sum_{j=1}^{n} \left(\frac{\partial p_j(y,s)}{\partial y_i} \frac{\partial x_j(y,s)}{\partial s} - \frac{\partial p_j(y,s)}{\partial s} \frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&\quad = \sum_{j=1}^{n} \frac{\partial p_j(y,s)}{\partial y_i} \frac{\partial F(p(y,s), z(y,s), x(y,s))}{\partial p_j} + \\
&+ \sum_{j=1}^{n} \left(\frac{\partial F(p(y,s), z(y,s), x(y,s))}{\partial x_j} + \frac{\partial F(p(y,s), z(y,s), x(y,s)) p_j(y,s)}{\partial z} \right) \frac{\partial x_j(y,s)}{\partial y_i} \\
&\quad = \frac{\partial}{\partial y_i} F(p(y,s), z(y,s), x(y,s)) - \\
&\quad - \frac{\partial F(p(y,s), z(y,s), x(y,s))}{\partial z} \left(\frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i} \right).
\end{aligned}$$

We insert the result $F(p(y,s),z(y,s),x(y,s))=0$ of the first step and obtain

$$\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y_i}(y,s)-\sum_{j=1}^{n}p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}\right)=$$
$$=-\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial z}\left(\frac{\partial z}{\partial y_i}(y,s)-\sum_{j=1}^{n}p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}\right).$$

This is a linear homogeneous ODE with initial value 0 at $s=0$. The unique solution vanishes identically. This implies the second equation and finishes the second step:

$$\frac{\partial z(y,s)}{\partial y_i}=\sum_{j=1}^{n}p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}.$$

Finally in a third step we show $p(y,s)=\nabla u(x(y,s))$ for all $(y,s)\in(\Omega\cap H)\times(-\epsilon,\epsilon)$. Locally the derivative of the map $(y,s)\mapsto x$ is invertible. Altogether we obtain

$$\frac{\partial u}{\partial x_j}=\frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j}+\sum_{i=1}^{n-1}\frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j}\quad=\left(\sum_{k=1}^{n}p_k\frac{\partial x_k}{\partial s}\right)\frac{\partial s}{\partial x_j}+\sum_{i=1}^{n-1}\left(\sum_{k=1}^{n}p_k\frac{\partial x_k}{\partial y_i}\right)\frac{\partial y_i}{\partial x_j}$$
$$=\sum_{k=1}^{n}p_k\left(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j}+\sum_{i=1}^{n-1}\frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right)\quad=\sum_{k=1}^{n}p_k\frac{\partial x_k}{\partial x_j}\quad=p_j.$$

Due to the initial values $z(y,0)$ we have $u(y)=g(y)$ for all $y\in\Omega\cap H$. Theorem 1.12 and the theorem of Picard-Lindelöf imply the uniqueness of the solutions. **q.e.d.**

We solved the boundary value problem by solving a family of ODEs. In the case of the inhomogeneous transport equation, we combine the coordinates x and t to one coordinate (x,t) and calculate the corresponding function F. Consequently we write

$$F(p,z,(x,t))=\tilde{F}(p,x,t)=b_1p_1+\ldots+b_np_n+p_{n+1}-f(x,t).$$

We use the equation $F(p,z,(x,t))=0$ and rewrite the ODE for z. Then the ODE becomes independent of p and we can solve $x(s)$, $t(s)$ and $z(s)$ separately:

$$x'(s)=b\qquad t'(s)=1,\qquad z'(s)=\tilde{F}(p(s),x(s),t(s))+f(x(s),t(s))=f(x(s),t(s)).$$

Whenever the function F is a first order polynomial with respect to p, then the functions

$$\frac{\partial F(p,z,x)}{\partial p_i}\text{ for }i=1,\ldots,n,\text{ and }F(p,z,x)-\sum_{j=1}^{n}\frac{\partial F(p,z,x)}{\partial p_j}p_j$$

do not depend on p. Therefore the ODE system becomes independent of $p(s)$, and the components $x(s)$ and $z(s)$ can be solved independently of $p(s)$. This situation describes the transport equation with vector b depending on z, x and t. For the solution of this equation we do not need to introduce the function $p(s) = \nabla u(x(s))$. Another example is the scalar conservation law in the general form for unknown function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$:

$$\dot{u}(x,t) + \nabla f(u(x,t)) = \dot{u}(x,t) + f'(u(x,t)) \cdot \nabla u(x,t) = 0$$

with a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}^n$. Again we impose the initial values $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}^n$ and some given function $u_0 : \mathbb{R}^n \to \mathbb{R}$. If $x_{n+1} = t$ then the corresponding function F is indeed linear in p:

$$F(p, z, (x,t)) = f'(z) \cdot (p_1, \ldots, p_m) + p_{n+1}.$$

So the corresponding ODE is independent of p

$$x'(s) = f'(z(s)) \qquad t'(s) = 1 \qquad z'(s) = F(p, z(s), (x(s), t(s))) = 0.$$

For any $x \in \mathbb{R}^n$ the unique solution is $x(s) = x + sf'(u_0(x))$, $t(s) = s$ and $z(s) = u_0(x)$. So we recover in this more general situation the implicit equation from Section 1.3:

$$u(x + tf'(u_0(x)), t) = u_0(x) \quad \text{for all} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$