Martin Schmidt
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Introduction to Partial Differential Equations
Exercise sheet 1

8th September, 2021

1. Multiindices and the Generalised Leibniz rule. In this question we introduce multiindex notation. A *multiindex* of n variables is a vector $\gamma \in \mathbb{N}_0^n$.

(a) Let $x = (x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$. Write out the full expression for the derivative $\partial^{(0,2,1)}$.

(b) Why do we need to assume that partial derivatives commute for multiindex notation to be useful?

(c) Which multiindices satisfy $|\gamma| \leq 2$ and which satisfy $\gamma \leq (0, 2, 1)$?

(d) The generalised binomial coefficient for multiindices is defined to be

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n}.$$

One justification for calling these binomial coefficients is the following property. Let $e_j = (0, \ldots, 1, \ldots, 0)$ be the multiindex with 1 is the j-th position and 0 in all other positions. Then for any j

$$\binom{\gamma}{\delta} = \binom{\gamma - e_j}{\delta - e_j} + \binom{\gamma - e_j}{\delta}.$$

Prove this property.

(e) Let $u, v : \Omega \to \mathbb{R}$ be smooth enough functions on an open subset $\Omega \subset \mathbb{R}^n$. Show for all multiindices $\gamma \in \mathbb{N}_0^n$ the following product rule:

$$\partial^\gamma(uv) = \sum_{0 \leq \delta \leq \gamma} \binom{\gamma}{\delta} \partial^\delta u \, \partial^{\gamma-\delta} v$$

2. Chain rule in multiple variables. Recall the chain rule for functions of multivariable variables (Satz 10.4(iii) in Schmidt's Analysis II script): Let $f : U \subset X \to Y$ be differentiable at $x_0 \in U$ and $g : V \subset Y \to Z$ be differentiable at $f(x_0) \in f[U] \subset V$. Then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

(a) Why does this chain rule above use function composition, when the chain rule for functions of a single variable uses multiplication? i.e.

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x. = 6x(x^2+1)^2.$$

(b) Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}^n$. Express the chain rule with partial derivatives to show that

$$\frac{d}{dt}u(x(t)) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{dx_i}{dt}.$$

(c) Write the above formula in terms of gradients and dot products.

(d) Consider the function $u(x, y) = x^2 + 2y$ and the polar coordinates $x = r\cos\theta, y = r\sin\theta$. Compute the radial and angular derivatives of u.

(e) Consider a scalar function $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of $2n + 1$ variables and a function $u : \mathbb{R}^n \to \mathbb{R}$. Write an expression for the derivative of $F(\nabla u(x), u(x), x)$ with respect to x_1.

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Introduction to Partial Differential Equations
Exercise sheet 2

14th September, 2021

3. Inhomogeneous Transport Equation. First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous equation

$$\frac{du}{dt} = f(t).$$

(a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

(b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique, if you haven't seen the lecture/read the script.) *(2 points)*

4. Royale with Cheese

Recall Burgers' equation from Example 1.6 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $u_0(x) = x$.

(a) According to Theorem 1.5, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point(s))*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 point(s))*

(c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution. *(2 point(s))*

(d) Finally, derive the following solution to the initial value problem:

$$u(x, t) = \frac{x}{1 + t}.$$

(2 point(s))

(e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 point(s))*

(f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point(s))*

5. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.7. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x,t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x,t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x,t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x,t)-plane for this function. *(2 point(s))*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(3 point(s))*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(3 point(s))*

Solutions are due on Monday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de` as a pdf. One possibility is to write your solutions neatly by hand and then scan them with your phone. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.

Martin Schmidt
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Introduction to Partial Differential Equations
Exercise sheet 3

21st September, 2021

6. Racecar go broooom.

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) What properties do you think that f should have? Does $f(u) = s_m u \cdot (1 - u/u_m)$ have these properties? *(2 point(s))*

(b) Find a function f that meets your conditions, or use the f from the previous part, and write down a PDE to describe the traffic flow. *(1 point(s))*

(c) Find all solutions that are constant in time. *(2 point(s))*

(d) Consider the situation of the start of a race: to the left of the starting line, the racecars are queued up at half of the maximum density (ie $0.5u_m$). To the right of the starting line, the road is empty. Now, at time $t = 0$, the race begins. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(5 point(s))*

7. All you can eat. Consider the the scalar conservation PDE for $f(u) = \frac{1}{3}u^3$ with the initial condition $u_0(x) = x$, similar to Exercise 4.

(a) Determine the characteristics of this problem. *(1 point(s))*

(b) Up until which time does there exist a strong solution? *(2 point(s))*

(c) Consider now the same PDE and initial condition, but with the domain $(t, x) \in [0, \infty) \times [0, \infty)$. Find the solution. Is it unique? *(4 point(s))*

(d) Consider now the same PDE and initial condition, but with the domain $(t, x) \in [0, \infty) \times [1, \infty)$. Find two solutions. *(3 point(s))*

8. Method of characteristics for an Inhomogeneous PDE Use the method of characteristics to solve the following *inhomogeneous* PDE. Note, the function u will *not* be constant along the characteristic, but its value along the characteristic will be determined by its initial value.

$$x\partial_x u + y\partial_y u = 2u$$

on the domain $x \in \mathbb{R}, y > 0$, with initial condition $u(x, 1) = x$.

(5 point(s))

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Introduction to Partial Differential Equations
Exercise sheet 4

28th September, 2021

9. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 point(s))

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.
. *(4 point(s))*

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(2 point(s))*

10. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $\partial_1 u + \partial_2 u = u^2$ on the plane with boundary condition $u(x_1, 0) = g(x_1)$.

(4 point(s))

(b) $x_1 \partial_2 u - x_2 \partial_1 u = u$ on the domain $x_2 > 0$, with boundary condition $u(0, x_2) = g(x_2)$.

(4 point(s))

(c) $u \partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2} x_1$.

(5 point(s))

11. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C xy \, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 bonus point(s))*

(b) Take $A = K = C$ in Definition 2.3. Consider the parametrisation of the circle $\Phi : (0, 2\pi)$, $t \mapsto (\cos t, \sin t)$. Compute the integral in this parametrisation. *(2 point(s))*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1 - x^2})$. Compute the integral in these parametrisations. *(2 point(s))*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. [Hint. The easiest way is to use two parametrisations similar to part (b)]

(e) Compute this integral using the divergence theorem. *(3 point(s))*

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Introduction to Partial Differential Equations
Exercise sheet 5

28th October, 2021

12. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the first Green formula

$$\int_\Omega v\triangle u \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v\nabla u \cdot N \, d\sigma.$$

(3 points)

(b) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v\triangle u - u\triangle v) \, dx = \int_{\partial\Omega} (v\nabla u - u\nabla v) \cdot N \, d\sigma.$$

(1 points)

(c) Suppose further that v has compact support in Ω. Prove that

$$\int_\Omega v\triangle u \, dx = \int_\Omega u\triangle v \, dx$$

(1 points)

13. The Black Hole.

Consider $\mathbb{R}^3$, a ball $B_r = \{x^2 + y^2 + z^2 \leq r^2\}$ and the function $g(x, y, z) = -(x^2 + y^2 + z^2)^{-0.5}$.

(a) Compute the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

where N is the outward pointing normal. Observe it that does not depend on the radius r.

(3 points)

(b) Can you apply the divergence theorem to this integral? Why or why not? *(1 point)*

(c) Compute the Laplacian of g. *(2 point)*

(d) Let $r < R$ and let $\Omega = B_R \setminus B_r$. The boundary of Ω has two components, namely ∂B_R and ∂B_r. Apply the divergence theorem to Ω with $f = \nabla g$. How does this relate to part (a)?

(3 points)

(e) Generalise the previous part to prove for any compact region $\Omega \subset \mathbb{R}^3$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \, d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

14. Convoluted.

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\, dy.$$

(a) Let $f_n(x) = n$ for $0 \le x \le n^{-1}$ and 0 otherwise. Show that the following bounds hold

$$\inf_{y \in I_n} g(y) \le (g * f_n)(0) \le \sup_{y \in I_n} g(y).$$

(3 Points)

(b) Suppose now that g is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n \to \infty$. *(3 Points)*

(c) (Optional) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation.

15. 🚀 🚀 🚀 🚀 🚀 🚀 🚀 🚀 🚀

In economics, the Black–Scholes equation is a PDE that describes the price V of a (European-style) option which under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 point(s))

16. Go with the flow.

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity ρv, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold S is the integral

$$\int_S \rho v \cdot N \, d\sigma,$$

where N is the normal of S.

(a) Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

(b) How does this relate to the form of the conservation law derived in the lectures?

(c) For liquids a common property is *incompressibility.* For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that ρ is constant. However, slightly more general model says that ρ is not globally constant, but if we follow a point $x(t)$ along the velocity field v then $\rho(x(t), t)$ is constant.

Use this description of incompressible flow to show that $\nabla \cdot v = 0$.

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Introduction to Partial Differential Equations
Exercise sheet 6

12th October, 2021

17. Distributions.

(a) Choose any compact set $K \subset \mathbb{R}$. Since it is bounded, there exists $R > 0$ with $K \subseteq [-R, R]$. Now choose any test function $\phi \in C_0^\infty(\mathbb{R})$ with compact support in K. Since it is continuous, $\sup_{x\in K} |\phi(x)|$ is finite. Prove the following inequality

$$\left| \int_0^\infty \phi(x)\, dx \right| \leq 2R \sup_{x\in K} |\phi(x)|$$

(2 Points)

(b) Show directly from Definition 2.6 that the Heaviside distribution

$$H : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_0^\infty \phi(x)\, dx$$

is a distribution on $\mathbb{R}$. *(2 Points)*

(c) Calculate and describe the first and second derivatives of the Heaviside distribution. *(3 Points)*

(d) Consider the differentiable function $f(x) = \sin x \in L^1_{\mathrm{loc}}(\mathbb{R})$. Recall the definition of the distribution F_f given prior to Lemma 2.9. Show that $(F_f)' = F_{f'}$ for this example. *(2 Points)*

(e) Consider the line $L = \{y = 1\} \subset \mathbb{R}^2$. Show that

$$G(\varphi) := \int_L \varphi\, d\sigma$$

defines a distribution in $\mathcal{D}'(\mathbb{R}^2)$. Note that the $d\sigma$ indicates this is an integration over the submanifold L. Does there exist a locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(\varphi) = \int_{\mathbb{R}^2} g\, \varphi\, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R})$? (Hint. Use Lemma 2.9) *(2 Points + 2 Bonus Points)*

18. Transport and Distribution.

We have seen that every distribution is differentiable. In this question we show that every distribution is also integrable. We use this to give a solution to the inhomogeneous transport equation for distributions. Throughout this question we consider test functions $\varphi(x, t)$ in $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ and distributions in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ unless stated otherwise. First we prepare some results about test functions. Some parts of this question are difficult, so try your best and don't be discouraged.

(a) Show that

$$(\mathcal{I}\varphi)(x) := \int_{\mathbb{R}} \varphi(x, t)\, dt$$

belongs to $C_0^\infty(\mathbb{R}^n)$. *(2 Points)*

(b) Show for any distribution $H \in \mathcal{D}'(\mathbb{R}^n)$ that $F : \varphi \mapsto H(\mathcal{I}\varphi)$ is a distribution in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. *(2 Points)*

(c) Define the subset $\mathcal{Z} = \ker \mathcal{I} = \{\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \mid \mathcal{I}\varphi \equiv 0\}$ and the operator

$$(\mathcal{P}\varphi)(x,t) := \int_{-\infty}^{t} \varphi(x,s)\, ds.$$

Show that $\mathcal{P}\varphi$ is a test function if and only if $\varphi \in \mathcal{Z}$. Moreover, show that if $\varphi \in \mathcal{Z}$ then $\mathcal{P}\varphi$ is the unique test function ψ with $\partial_t \psi = \varphi$. *(3 Points)*

(d) Let χ_0 be a test function that does not depend on x with $\int_{\mathbb{R}} \chi_0(t)\, dt = 1$. For any test function φ, define

$$\tilde{\varphi}(x,t) := \varphi(x,t) - (\mathcal{I}\varphi)(x)\chi_0(t).$$

Show that $\tilde{\varphi} \in \mathcal{Z}$. *(1 Point)*

Part (d) gives a decomposition of any test function into a derivative of a test function and the product of test functions that are constant in t and x. Now we are ready to show that distributions can be 'integrated' with respect to t.

(e) Suppose that U and F are two distributions such that $\partial_t U = F$. Why must $U(\partial_t \varphi) = -F(\varphi)$ for any test function φ? *(1 Point)*

(f) Suppose we are given a distribution F. Prove that, for any $G \in \mathcal{D}'(\mathbb{R}^n)$, the following formula defines a distribution such that $\partial_t U = F$: *(1 Point + 2 Bonus Points)*

$$U(\varphi) := -F(\mathcal{P}\tilde{\varphi}) + G(\mathcal{I}\varphi).$$

Thus we see that every distribution F has many t-antiderivatives. The distribution G plays the role of the integration constant, in the same way that for ordinary functions $\partial_t(f(x,t) + g(x)) = \partial_t f$. It turns out that the converse is also true, every t-antiderivative of F has this form. This is essentially proved in Exercise 2.10(3) from the lecture script.

Finally, we show that the inhomogeneous transport equation for distributions $\partial_t U + b \cdot \nabla U = F$ is always solvable.

(g) Let $(\mathcal{T}_b\varphi)(x,t) := \varphi(x - bt, t)$ be the translation operator. For any distribution U define a distribution $\tilde{U} : \varphi \mapsto U(\mathcal{T}_b\varphi)$. Notice that although $\tilde{U}(\varphi) = U(\mathcal{T}_b\varphi)$ for any test function φ, their derivatives are subtly distinct in a way that can be hard to express in notation, namely

$$\partial_i U(\mathcal{T}_b\varphi) = -U(\partial_i(\mathcal{T}_b\varphi)), \qquad \partial_i \tilde{U}(\varphi) = -\tilde{U}(\partial_i\varphi) = -U(\mathcal{T}_b(\partial_i\varphi)).$$

Prove that $\partial_t \tilde{U}(\varphi) = \partial_t U(\mathcal{T}_b\varphi) + b \cdot \nabla U(\mathcal{T}_b\varphi)$. *(2 Points)*

(h) Finally, let F be any distribution. Use parts (f) and (g) to give a solution to the inhomogeneous transport equation. *(2 Bonus Points)*

We have not addressed the question of uniqueness; this is ultimate aim of Exercise 2.10 in the lecture script, which finds every solution to the homogeneous equation. Since the difference of two solutions to the inhomogeneous equation is a solution to the homogeneous equation, this suffices.

Neither this question or 2.10 really shows you what it means to specify the 'initial value' of a distributional PDE, though it is hinted at by the G above and 2.10(4). I leave this as a challenge to you. But if we contrast the solutions we have found here to the results of Section 1 (there is a solution for every *differentiable* function g), we see that distributions allows us to have non-differentiable solutions to PDEs in a rigorous way.

19. Preparing the Mean Value Theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$, and $\partial B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$ for $r > 0$. Show that the function

$$F(r) := \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) \,\mathrm{d}\sigma(x)$$

converges to $f(x_0)$ as $r \to 0$. *(4 Points)*

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Introduction to Partial Differential Equations
Exercise sheet 7

18th October, 2021

20. Twirling towards freedom.

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function.

(a) Show that the following functions are also harmonic.

(i) $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$.

(ii) $v(x) = u(ax)$ for $a \in \mathbb{R}$.

(iii) $v(x) = u(Rx)$ for $R(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ the reflection operator.

(iv) $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under all Euclidean motions and harmonic functions can be rescaled. *(5 Points)*

(b) Show, using the chain rule, that in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ the Laplacian is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(3 Points)

(c) Hence show that $v(r, \theta) = u(r^{-1}, \theta)$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$. *(2 Points)*

21. Harmonic Polynomials in Two Variables.

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 Point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x, y) = (x + \iota y)^n$ and let $u_n(x, y)$ and $v_n(x, y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(2 Points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n-1$. *(1 Point)*

(d) Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(3 Bonus Points)*

22. Means and Ends

In the lecture script we often encounter the *spherical mean* of v:

$$\Phi(v, x, r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v(y) \, \mathrm{d}\sigma(y).$$

We have seen in a previous exercise that $\lim_{r\to 0} \Phi(v, x, r) = v(x)$ when v is continuous. Let $v \in C^2(\overline{\Omega})$ be any twice continuously differentiable function. Carefully justify the formula

$$\frac{\partial}{\partial r}\Phi(v, x, r) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta v(x_0 + rz) \, \mathrm{d}z.$$

This formula is used in the proof of the Mean Value property. It shows why spherical means and harmonic functions are related. *(5 Points)*

23. Liouville's Theorem.

Let $u \in C^2(\mathbb{R}^2)$ be a harmonic function. Liouville's theorem (3.5 in the script) says that if u is bounded, then u is constant. In this question we give a geometric proof using *ball means*. Similar to a spherical mean, the ball mean of a function $v \in C(\overline{\Omega})$ is defined when $\overline{B(x,r)} \subset \Omega$:

$$M(v,x,r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y)\, \mathrm{d}y$$

This proof comes from the following article Nelson, 1961.

(a) Show that u obeys the mean value property on balls, $u(x) = M(u,x,r)$.
(Hint. use polar coordinates for the integral $\mathrm{d}y = \mathrm{d}\sigma\, \mathrm{d}\rho$.) *(2 Points)*

(b) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 Bonus Points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(c) Suppose that u is bounded on the plane: $-C \leq u(x) \leq C$ for all x and some constant C. Show that *(2 Points)*

$$\left|M(u,a,r) - M(u,b,r)\right| \leq \frac{2C}{\omega_2}\left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right)$$

(d) Complete the proof that u is constant. *(2 Points)*

Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 8

25th October, 2021

24. Back in the saddle.

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(2 Points)*

25. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean: $v(x) \leq \Phi(v, x, r)$.

(a) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 Points)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω. *(3 Points)*

(c) Let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that $\|\nabla u\|^2$ is subharmonic. *(2 Points)*

(d) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic. *(1 Point)*

26. Never judge a book by its cover.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(4 Points)*

27. Weak Tea.

Choose any point $x \in \mathbb{R}^n$. For each function $\psi \in C_0^\infty((0,r))$ there is a test function

$$f_{x,\psi}(y) \in C_0^\infty(B(x,r)), \qquad f_{x,\psi(y)} = \begin{cases} \frac{\psi(|y-x|)}{n\omega_n\,|y-x|^{n-1}} & \text{for } y \neq x \\ 0 & \text{otherwise} \end{cases}$$

given in the definition of the weak mean value property 3.6. We try in this question to develop some intuition for these functions.

(a) Justify that $f_{x,\psi}$ is indeed a test function. *(2 Points)*

(b) For the delta distribution δ_x centred at x, compute $\delta_x(f_{x,\psi})$. *(1 Point)*

(c) Does δ_x satisfy the weak mean value property? Defend your answer. *(2 Points)*

(d) Let λ_ε be a mollifier on $\mathbb{R}$ and define $\psi_\varepsilon(r) = \lambda_\varepsilon(r-1)$. For $\varepsilon < 1$ this obeys $\psi_\varepsilon \in C_0^\infty((1-\varepsilon, 1+\varepsilon)) \subset C_0^\infty((0,2))$. Let $f_\varepsilon = f_{0,\psi_\varepsilon}$ be the corresponding test functions for balls centred at the origin. Let g be a continuous function. Show that as $\varepsilon \to 0$

$$F_g(f_\varepsilon) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{g(y)}{|y|^{n-1}} \, \psi_\varepsilon(|y|) \, dy \to \Phi(g, 0, 1).$$

(Hint. Lemma 2.8.) *(5 Points)*

Martin Schmidt
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Introduction to Partial Differential Equations
Exercise sheet 9

1st November, 2021

28. Do nothing by halves.

Let $H_1^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \dots, x_n)$ reflection in the plane H^0. Let $Q_{12} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0, x_2 > 0\}$ be a quadrant.

(a) A reflection principle for harmonic functions. Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0. Show that the function $v : \mathbb{R}^n \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic. *(4 Points)*

(b) Green's function for the half-space. Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

(3 Points)

(c) Green's function for the quadrant. Compute the Green's function for Q_{12}. *(3 Points)*

29. Teach a man to fish...

(a) Using the Green's function of H_1^+ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem $\Delta u = 0$ in $H_1^+, u|_{H_1^0} = g$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x - z|^n} \, \mathrm{d}\sigma(z)$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined. *(3 Points)*

(b) Show that if g is periodic (that is, there is some vector $L \in \mathbb{R}^{n-1}$ with $g(x + L) = x$ for all $x \in \mathbb{R}^{n-1}$) then so is the solution. *(2 Points)*

(c) Now consider the plane $n = 2$ with g function with compact support. Approximate the value of $u(x)$ for large $|x|$. Feel free to modify this question as you see fit, what interesting things can you say about the growth of u?

(2 Points + Bonus Points as deserved)

30. One of a kind.

Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected domain and $u, v \in C^2(\overline{\Omega})$ two harmonic functions. Suppose that there is an open subset $U \subset \Omega$ such that $u = v$ on U. Prove that $u = v$ on Ω using Corollary 3.22 (Harmonic functions are analytic). This is called the *unique continuation property* of harmonic functions. *(2 Points)*

31. To be or not to be.

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

(a) Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.24. *(2 Points)*

(b) Generalise this: What are the necessary and sufficient conditions on g for the Dirichlet problem to have a solution on this domain? *(2 Points)*

(c) Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points? *(1 Bonus Point)*

Martin Schmidt
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Introduction to Partial Differential Equations
Exercise sheet 10

8th November, 2021

32. Special solutions of the heat equation.

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 Point)*

(b) Look for "separable" solutions of the heat equation: those of the form $u(x,t) = X(x)T(t)$. Argue that there is constant λ such that

$$\dot{T}(t) = -\lambda T(t), \qquad -\Delta X(x) = \lambda X(x),$$

for all x and t. *(2 Points)*

(c) Suppose that Ω is a bounded domain, and that $u|_{\partial\Omega} = 0$. Apply the Green's first formula with $v = u$ to an eigenfunction of Δ to show that λ can only be positive. *(2 Points)*

(d) How do separable solutions behave over time? *(1 Point)*

33. Geothermal Power.

Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x,0) = h(x)$. Suppose, as an ansatz, that the solution is a power series in t, i.e. $u(x,t) = \sum_{k=0}^{\infty} a_k(x)\, t^k$ for functions $a_k : \mathbb{R}^n \to \mathbb{R}$.

(a) Why is $a_0 = h$? *(1 Point)*

(b) Show that the a_k obey the recursion relation $a_{k+1} = \frac{1}{k+1}\Delta a_k$. *(2 Points)*

(c) Hence conclude that $u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}(\Delta^k h)(x) t^k$. *(1 Point)*

(d) Suggest some conditions on h that would ensure this series converges. *(1 Bonus Point)*

34. The distribution of heat.

Consider the fundamental solution of the heat equation $\Phi(x,t)$ given in Definition 4.1.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 Points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 Points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x,t)\varphi(x,t)\; dx\, dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)\; dx\, dt \;\; ?$$

(1 Point)

(d) Conclude with the help of Lemma 4.2 and Theorem 4.3 that

$$|H(\varphi)| \leq T\,\|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(3 Points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.3 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\;dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 Point)*

(f) Hence show that

$$\int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\;dy\,dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(4 Points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_0^{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\;dy\,dt \to 0$$

(2 Points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{\infty}\right) \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\;dy\,dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.

Martin Schmidt
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Introduction to Partial Differential Equations
Exercise sheet 11

15th November, 2021

35. Sugar, we're going down swinging.

First let $\Omega' \subset \mathbb{R}^n \times \mathbb{R}$ be an open and connected region. A function $v : \Omega' \to \mathbb{R}$ is called a *sub-solution* of the heat equation if $\dot{v} - \Delta v \leq 0$.

(a) *Mean value estimate for sub-solutions* Take any point $(x, t) \in \Omega'$ and a small radius $r > 0$ so that $E(x, t, r) \in \Omega'$ (refer to Definition 4.6). Modify the proof the mean value property of the heat equation to show that

$$v(x,t) \leq \frac{1}{4r^n} \int_{E(x,t,r)} v(y,s) \frac{|x-y|^2}{|t-s|^2} \, d^n y \, ds$$

holds for all sub-solutions. *(2 Points)*

Now let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and path connected region. We denote the parabolic cylinder of Ω by $\Omega_T := \Omega \times (0, T]$ as in Section 4.4. Suppose that $v : \Omega_T \to \mathbb{R}$ is a sub-solution that extends continuously to $\overline{\Omega_T}$.

(b) *Maximum principle for sub-solutions* Following on from (a), establish that if v takes the value $\sup_{\Omega_T} v$ on Ω_T, then it is constant. *(2 Points)*

(c) *A monotonicity property* For $j \in \{1, 2\}$ let $f_j : \Omega \times (0, T) \to \mathbb{R}$, $h_j : \Omega \to \mathbb{R}$, and $g_j : \partial\Omega \times [0, T]$ be smooth functions, and likewise let $u_j : \Omega \times (0, T)$ be smooth functions with continuous extensions to the boundary that satisfy

$$\begin{cases} \dot{u}_j - \Delta u_j = f_j & \text{on } \Omega \times (0, T) \\ u_j(x, 0) = h_j(x) & \text{on } \Omega \\ u_j = g_j & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

Suppose further that $f_1 \leq f_2$, $g_1 \leq g_2$, and $h_1 \leq h_2$. Show in this case that $u_1 \leq u_2$ as well. *(2 Points)*

36. Heat death of the universe.

First a corollary to Theorem 4.3:

(a) Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.3. Show

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}} \|h\|_{L^1}.$$

(2 Points)

The above corollary shows how solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ with such initial conditions behave: they tend to zero as $t \to \infty$. Physically this is because if $h \in L^1$ then there is a finite amount of total heat, which over time becomes evenly spread across the plane.

On open and bounded domains $\Omega \subset \mathbb{R}^n$ we can have different behaviour, due to the boundary conditions holding the temperature steady. In this question we determine the long time behaviour of solutions u to the heat equation on open and bounded sets Ω with $u(x,t) = g(x)$ on $\partial\Omega \times \mathbb{R}^+$ and $u(x,0) = h(x)$. Assume that there is a steady state solution, i.e. a solution to the Dirichlet problem for the Laplace equation $\Delta v = 0$ and $v|_{\partial\Omega} = g$. We claim $u \to v$ as $t \to \infty$. Let $w(x,t) = u(x,t) - v(x)$. The claim is equivalent to $w \to 0$.

(b) What PDE and boundary conditions does w obey? *(2 Points)*

(c) Let l_m be the function from Theorem 4.3 that solves heat equation on $\mathbb{R}^n$ with $l_m(x,0) = mk(x)$ for m a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that $k|_\Omega \equiv 1$. Why must k exist? Why does $l_m \to 0$ as $t \to \infty$? What boundary conditions on Ω does it obey? *(3 Points)*

(d) Use the monotonicity property to show that w tends to zero. *(2 Points)*
Hint. Consider $a = \sup_{x\in\Omega} |w(x,0)|$.

37. The Fourier transform.

Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined in Section 4.6 to be a function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y} h(y)\, dy.$$

Lemma 4.20 shows that it is well-defined for Schwartz functions.

(a) Give the definition of a Schwartz function. *(1 Point)*

(b) Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function. *(1 Point)*

(c) Show that the Fourier transform of $\exp(-A^2x^2)$ for a constant $A > 0$ is $\sqrt{\pi}A^{-1}\exp(-\pi^2k^2A^{-2})$. You may use that $\int_{\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$. *(2 Points)*

(d) Show that $\widehat{\partial_j f}(k) = 2\pi i k_j \hat{f}(k)$ for Schwartz functions $f : \mathbb{R}^n \to \mathbb{R}$. *(2 Points)*

(e) If $u : \mathbb{R} \times \mathbb{R}^+$ is a solution to the heat equation, we can apply a Fourier transform in the space coordinate to get a function $\hat{u}(k,t)$. Show that this function obeys

$$\frac{\partial \hat{u}}{\partial t} + 4\pi^2 k^2 \hat{u} = 0.$$

Solve this ODE in the time variable. *(2 Points)*

(f) Suppose that we have the initial condition $u(x,0) = h(x)$ for $x \in \mathbb{R}$ for a Schwartz function h. Then $\hat{u}(k,0) = \hat{h}(k)$. Apply the inverse Fourier transformation to rederive the solution given in Theorem 4.3. *(2 Points)*

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Introduction to Partial Differential Equations
Exercise sheet 11

22nd November, 2021

38. One step at a time.

Prove the following identity for the heat kernel in one dimension ($n = 1$):

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s)\, dy.$$

Interpret this equation in the context of the heat equation on the line. *(4 Points)*

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$

39. Some like it hot.

Find the solution $u : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem using the heat kernel of $[0,1]$:

$$\begin{cases} \dot{u} - 3\partial_{xx} u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = \sin(\pi x) + 2\sin(5\pi x) & \text{for } x \in (0,1). \end{cases}$$

(6 Points)

40. The heat kernel on $\mathbb{S}^1$.

(Adapted from Exercise 4.22 in the lecture script) Denote the fundamental solution of the heat equation by $\Phi(x,t)$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Show that

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

defines a smooth periodic function with period 1 (i.e. $\tilde{f}(x+1) = \tilde{f}(x)$). *(2 Points)*

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, with period 1, and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ a solution to the heat equation with initial condition $u(x,0) = h(x)$. Show that u remains periodic in the spatial coordinate for all time. *(2 Points)*

(c) Conclude that

$$u(x,t) := \int_{\mathbb{S}^1} h(y) \sum_{n\in\mathbb{Z}} \Phi(x-y+n, t).$$

solves the heat equation with the initial condition. *(2 Points)*

(d) Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}.$$

Show, with the aid of this equality, the relation

$$\Theta(x-y, 4\pi it) = \sum_{n\in\mathbb{Z}} \Phi(x-y+n, t),$$

where the left hand side is the Jacobi's theta function from Section 4.7.

(2 Points)

(e) How would you modify Definition 4.14 to give give an abstract definition of the heat kernel $H_{\mathbb{S}^1}$? *(2 Points)*

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Introduction to Partial Differential Equations
Exercise sheet 13

29th November, 2021

41. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = 0 \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = 1$ and $h(x) = \cos x$. *(3+2 Points)*

42. Waves and Distributions.

Recall from Analysis the definition of the directional derivative D_v in the direction $v \in \mathbb{R}^n$ is given by

$$D_v f = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}.$$

The partial derivatives are special cases of the directional derivative, and for a function that is differentiable we know that $D_v f = v \cdot \nabla f$.

(a) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $(x,t) \mapsto F(x + t)$ is differentiable with respect to $D_{(1,-1)}$. In fact, show that this directional derivative is zero. (Note, F might not be differentiable and this doesn't matter.) *(2 Points)*

(b) Explain why, for smooth functions, $\partial_t^2 - \partial_x^2 = D_{(1,-1)} D_{(1,1)}$. *(2 Points)*

(c) Consider the distribution $\tilde{F}$ given by $\tilde{F}(\varphi) = \int_{\mathbb{R}^2} F(x + t)\varphi(x,t)\ dx\, dt$. (Optional: show this is a distribution.) Show that it solves the wave equation $(\partial_t^2 - \partial_x^2)\tilde{F} = 0$ in the sense of distributions. *(2 Points)*

One can apply the same reasoning to show that $F(x - t)$ also solves the wave equation in the weak sense. It is for this reason we can claim that the general solution to the wave equation in one-dimension is $F(x+t)+G(x-t)$ for any continuous functions F and G. To extend this result to all distributions, it's best to proceed with the translation operator from Sheet 6 Question 18(g) and those sort of methods.

43. Represent.

In this question we derive a representation formula for the one dimensional wave equation, similar to the Poisson representation formula and Corollary 4.5. The difference is that we will try to do as much of the proof as we can using distributions and convolution algebra and avoiding explicit integrals.

We have not proved all the results needed to give a rigorous proof, but I think they are believable. Namely, you are free to assume the following two facts:
(i) the Leibniz rule holds for the derivative of a product of a smooth function and a distribution $\partial_i(gF) = (\partial_i g)F + g(\partial_i F)$, and

(ii) the derivative property of convolutions also holds for distributions $\partial_i(g * F) = (\partial_i g) * F = g * (\partial_i F)$.

Let χ_A be the characteristic function for the set A. That means $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise.

(a) Let $K : \mathbb{R}^2 \to \mathbb{R}$ be defined as $K(x,t) = \frac{1}{2}\chi_{\{t\geq 0\}}\chi_{\{-t\leq x\leq t\}}$. Show that $(\partial_t^2 - \partial_x^2)K = \delta$ in the sense of distributions. That is, show for all test functions $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ that

$$\int_0^\infty \int_{-t}^t \frac{1}{2}(\partial_t^2 - \partial_x^2)\varphi(x,t)\, dx\, dt = \varphi(0,0).$$

This shows that K is a fundamental solution of the wave equation. *(3 Points)*

(b) Suppose that $u : \mathbb{R} \times [0,\infty)$ is a (smooth) solution to the inhomogeneous wave equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x). \end{cases}$$

We can extend this to the whole plane by considering $u\chi_H$ for $H := \mathbb{R} \times [0,\infty)$ and likewise for the functions f, g, h. Explain and/or add further steps of working to the following calculation: For $t > 0$

$$\begin{aligned} u(x,t) &= \delta * u\chi_H && (1)\\ &= \partial_t^2 K * u\chi_H - \partial_x^2 K * u\chi_H && (2)\\ &= \partial_t^2 K * u\chi_H - K * (\partial_t^2 u)\chi_H + K * f\chi_H. && (3) \end{aligned}$$

(5 Points)

(c) Explain and/or add further steps of working to the following calculation, including the correct definition of $\delta_{t=0}$.

$$\begin{aligned} \partial_t^2 K * u\chi_H &= \partial_t K * (\partial_t u)\chi_H + \partial_t K * u\delta_{t=0} && (4)\\ &= K * (\partial_t^2 u)\chi_H + K * (\partial_t u)\delta_{t=0} + \partial_t(K * u\delta_{t=0}). && (5) \end{aligned}$$

(3 Points)

(d) Hence write down a representation formula for u in terms of the given data f, g, h, similar to the others in this course. *(3 Points)*

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Introduction to Partial Differential Equations
Exercise sheet 14

6th December, 2021

44. Method of Descent

In this exercise we will apply the method of descent to solve the wave equation on $\mathbb{R}^2$ for a particular set of initial conditions. The idea is to help you understand the key ideas and notation of the method. It is a combination of results from Sections 5.1–4.

Consider the wave equation on $\mathbb{R}^2$ with initial conditions

$$\begin{aligned} &\partial_t^2 u - \Delta u = 0 \text{ on } (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ &u(x,0) = g(x) = \chi_{[0,\infty)}(x_1), \qquad \partial_t u(x,0) = h(x) = 0. \end{aligned}$$

(a) In Section 5.4 we define $\bar{u} : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ associated to the function u. Explain why $\bar{g}(x_1,x_2,x_3) = \chi_{[0,\infty)}(x_1)$ and $\bar{h} = 0$ (or give the definition of bar). Why does $\bar{u}$ solve the wave equation on $\mathbb{R}^3$? (note, the Laplacians are different in different dimensions). *(2 Points)*

(b) Conversely, prove that a solution $\bar{u}$ to the 3-dimensional wave equation that does not depend on x_3 gives a solution to the 2-dimensional wave equation. *(2 Points)*

(c) By (a) and (b), we now must solve a wave equation on $\mathbb{R}^3$. The key to solving the 3-dimensional wave equation is to consider the (spatial-)spherical means

$$U(x,t,r) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \bar{u}(z,t)\, d\sigma(z),$$

and likewise let G and H be the spherical means of $\bar{g}$ and $\bar{h}$ respectively. Show that

$$G(x,r) = \begin{cases} 0 & \text{for } x_1 \leq -r \\ \frac{1}{2}\frac{x_1+r}{r} & \text{for } |x_1| \leq r \\ 1 & \text{for } r \leq x_1 \end{cases} \qquad \text{and } H(x,r) = 0.$$

You may use the following geometric fact: for $-R < a < b < R$, the surface area of the part of the sphere $\partial B(0,R)$ with $a < x_1 < b$ is $2\pi R(b-a)$. *(4 Points)*

(d) We know by Lemma 5.2 that U obeys the Euler-Poisson-Darboux equation. Let $\tilde{U}(x,t,r) := rU(x,t,r)$. Show that $\tilde{U}$ obeys the following PDE

$$\begin{aligned} &\partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 \text{ on } (t,r) \in [0,\infty) \times [0,\infty), \\ &\tilde{U}(x,0,r) = rG(x,r), \qquad \partial_t \tilde{U}(x,0,r) = rH(x,r). \end{aligned}$$

Note that x plays no role in this PDE, so we can think of it as a family of PDEs parametrised by x. *(2 Points)*

(e) Thus we see that $\tilde{U}$ obeys the 1-dimensional wave equation on the *half-line* $r \in [0,\infty)$. This is solved by a trick using reflection, and the formula is at the end of Section 5.1. We only need the solution for small r, so it is enough to consider the case $0 \le r \le t$. In this case, show

$$\tilde{U}(x,t,r) = \begin{cases} 0 & \text{for } x_1 \le -(t+r) \\ \frac{1}{4}(x_1+t+r) & \text{for } -(t+r) \le x_1 \le -(t-r) \\ \frac{1}{2}r & \text{for } |x_1| \le t-r \\ \frac{1}{4}(x_1-t+3r) & \text{for } t-r \le x_1 \le t+r \\ r & \text{for } x_1 \ge t+r. \end{cases}$$

(4 Points)

(f) Recover $\bar{u}$ from $\tilde{U}$ using a certain property of spherical means. *(1 Point + 2 Bonus Points)*

Observe that $\bar{u}$ does not depend on x_3. So by part (b) we have a solution to the 2-dimensional wave equation:

$$u(x_1,x_2,t) = \begin{cases} 0 & \text{for } x_1 < -t \\ 0.25 & \text{for } x_1 = -t \\ 0.5 & \text{for } -t < x_1 < t \\ 0.75 & \text{for } x_1 = t \\ 1 & \text{for } x_1 > t. \end{cases}$$

This solution has jump discontinuities, but this is unsurprising since the initial conditions also had them.

45. Plane Waves.

Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a solution to the following modified wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} c_j^2 \frac{\partial^2 u}{\partial x_j^2} = 0\ , \qquad (*)$$

where $c_1, \ldots, c_n > 0$ are constants.

(a) Let $\alpha \in \mathbb{R}^n$ be a unit vector $\|\alpha\| = 1$, $\mu \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable function. Show that

$$u(x,t) := F(\alpha \cdot x - \mu t)$$

is a solution of $(*)$ exactly when

$$\mu^2 = \sum_{j=1}^{n} \alpha_j^2 c_j^2$$

or F is linear. Solutions of $(*)$ with this form are called *plane waves.* *(2 Points)*

(b) For the solutions in (a), examine whether the following property holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$u(x,t) = u(x - \mu t\alpha, 0).$$

Interpret this equation in terms of direction and speed. *(3 Points)*

46. Electromagnetic Waves.

In physics, electrical and magnetic fields are modelled as time-dependent vector fields, which mathematically are smooth functions $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. Through a series of experiments in the 18th and 19th centuries, the existence and properties of these fields were discovered. Importantly, it was discovered that the two phenomena were connected (both magnets and static electricity had been known since antiquity). In 1861 James Clerk Maxwell published a series of papers summarising electromagnetic theory, including a collection of 20 differential equations. Over time these were further reduced to the following four (by Heaviside 1884 using vector notation), called *Maxwell's Equations*:

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

As is usual, the ∇ operator acts on the spatial coordinates x, and the $\times$ denotes the cross product of $\mathbb{R}^3$. The constants ε_0, the electrical permittivity, and μ_0, the magnetic permeability, are approximately $\varepsilon_0 \approx 8{,}854 \cdot 10^{-12} \frac{\mathrm{A \cdot s}}{\mathrm{V \cdot m}}$ and $\mu_0 \approx 1{,}257 \cdot 10^{-6} \frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}$ (V=Volt, s=Seconds, A=Ampere and m=Metre) in a vacuum. Electrical charges are included via the charge density ρ and electric currents are the movements of charges, $J := v\rho$ for a velocity field v.

The two equations with divergence were formulated by Gauss, the curl of the electric field is due to Faraday, and the curl of the magnetic field is due to Ampère. The last term in Ampère's law that has the time-derivative of the electrical field was an addition of Maxwell. With this correction, he was able to derive the equations for electromagnetic waves, as you will now do.

(a) Let E und B be solutions to Maxwell's equations in the absence of electric charges, $\rho = 0, J = 0$. Show that they each satisfy a modified wave equation (Question 41). You may use without proof the identity $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \triangle f$ for smooth functions $f : \mathbb{R}^3 \to \mathbb{R}^3$. *(3 Bonus Points)*

(b) Predict the speed of these waves. *(2 Bonus Points)*

(c) Argue that Ampère's law in its original form $\nabla \times B = \mu_0 J$ violates the conservation of charge ρ under some conditions. Refer to Exercise Sheet 5 Question 16 for the definition of a conservation law. Thereby derive Maxwell's additional term. *(3 Bonus Points)*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.