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Introduction to Partial Differential Equations
Exercise sheet 13

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41. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = 0 \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = 1$ and $h(x) = \cos x$. *(3+2 Points)*

Solution. One can rescale one of the coordinates to compensate for the factor of a^2. Namely, let $\tau = at$. Because $t = 0$ when $\tau = 0$, the first initial condition is unchanged. The second initial condition however reads $a\partial_\tau u(x,0) = h(x)$. Using the formula for the solution to this new initial value problem for the wave equation, but then further making the substitution $\tau = at$, gives

$$u(x,t) = \frac{1}{2}\left[g(x+at) + g(x-at)\right] + \frac{1}{2}\int_{x-at}^{x+at} \frac{1}{a} h(y)\, dy.$$

With the given initial data

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[1+1\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \cos y\, dy \\ &= 1 + \frac{1}{4}\left[\sin(x+2t) - \sin(x-2t)\right]. \end{aligned}$$

42. Waves and Distributions.

Recall from Analysis the definition of the directional derivative D_v in the direction $v \in \mathbb{R}^n$ is given by

$$D_v f = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}.$$

The partial derivatives are special cases of the directional derivative, and for a function that is differentiable we know that $D_v f = v \cdot \nabla f$.

(a) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $(x,t) \mapsto F(x+t)$ is differentiable with respect to $D_{(1,-1)}$. In fact, show that this directional derivative is zero. (Note, F might not be differentiable and this doesn't matter.) *(2 Points)*

(b) Explain why, for smooth functions, $\partial_t^2 - \partial_x^2 = D_{(1,-1)}D_{(1,1)}$. *(2 Points)*

(c) Consider the distribution $\tilde{F}$ given by $\tilde{F}(\varphi) = \int_{\mathbb{R}^2} F(x+t)\varphi(x,t)\, dx\, dt$. (Optional: show this is a distribution.) Show that it solves the wave equation $(\partial_t^2 - \partial_x^2)\tilde{F} = 0$ in the sense of distributions. *(2 Points)*

One can apply the same reasoning to show that $F(x-t)$ also solves the wave equation in the weak sense. It is for this reason we can claim that the general solution to the wave equation in one-dimension is $F(x+t)+G(x-t)$ for any continuous functions F and G. To extend this result to all distributions, it's best to proceed with the translation operator from Sheet 6 Question 18(g) and those sort of methods.

Solution.

(a) By direct calculation

$$D_{(1,-1)}F = \lim_{h\to 0} \frac{F((x+h)+(t-h)) - F(x+t)}{h} = \lim_{h\to 0} 0.$$

Thus we see the limit exists and is zero at every point (x,t).

(b) For smooth functions, we know that the directional derivative can be expressed as partial derivatives. Indeed $D_{(1,-1)} = (1,-1)\cdot(\partial_x, \partial_t) = \partial_x - \partial_t$ and $D_{(1,1)} = \partial_x + \partial_t$. So

$$D_{(1,-1)}D_{(1,1)} = (\partial_x - \partial_t)(\partial_x + \partial_t) = \partial_x^2 + \partial_x\partial_t - \partial_t\partial_x - \partial_t^2 = \partial_x^2 - \partial_t^2.$$

(c) The essential point is that there is nothing special about partial derivatives when it comes to integration by parts, it also holds for any directional derivative. So

$$\begin{aligned}(\partial_t^2 - \partial_x^2)\tilde{F}(\varphi) = \tilde{F}((\partial_t^2 - \partial_x^2)\varphi) &= \int_{\mathbb{R}^2} F(x+t)\, D_{(1,-1)}D_{(1,1)}\varphi \; d\mu \\ &= -\int_{\mathbb{R}^2} D_{(1,-1)}(F(x+t))\, D_{(1,1)}\varphi \; d\mu = 0.\end{aligned}$$

43. Represent.

In this question we derive a representation formula for the one dimensional wave equation, similar to the Poisson representation formula and Corollary 4.5. The difference is that we will try to do as much of the proof as we can using distributions and convolution algebra and avoiding explicit integrals.

We have not proved all the results needed to give a rigorous proof, but I think they are believable. Namely, you are free to assume the following two facts:
(i) the Leibniz rule holds for the derivative of a product of a smooth function and a distribution $\partial_i(gF) = (\partial_i g)F + g(\partial_i F)$, and
(ii) the derivative property of convolutions also holds for distributions $\partial_i(g * F) = (\partial_i g) * F = g * (\partial_i F)$.

Let χ_A be the characteristic function for the set A. That means $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise.

(a) Let $K : \mathbb{R}^2 \to \mathbb{R}$ be defined as $K(x,t) = \frac{1}{2}\chi_{\{t\geq 0\}}\chi_{\{-t\leq x\leq t\}}$. Show that $(\partial_t^2 - \partial_x^2)K = \delta$ in the sense of distributions. That is, show for all test functions $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ that

$$\int_0^\infty \int_{-t}^{t} \frac{1}{2}(\partial_t^2 - \partial_x^2)\varphi(x,t)\; dx\, dt = \varphi(0,0).$$

This shows that K is a fundamental solution of the wave equation. *(3 Points)*

(b) Suppose that $u : \mathbb{R} \times [0, \infty)$ is a (smooth) solution to the inhomogeneous wave equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x). \end{cases}$$

We can extend this to the whole plane by considering $u\chi_H$ for $H := \mathbb{R} \times [0, \infty)$ and likewise for the functions f, g, h. Explain and/or add further steps of working to the following calculation: For $t > 0$

$$\begin{aligned} u(x, t) &= \delta * u\chi_H && (1) \\ &= \partial_t^2 K * u\chi_H - \partial_x^2 K * u\chi_H && (2) \\ &= \partial_t^2 K * u\chi_H - K * (\partial_t^2 u)\chi_H + K * f\chi_H. && (3) \end{aligned}$$

(5 Points)

(c) Explain and/or add further steps of working to the following calculation, including the correct definition of $\delta_{t=0}$.

$$\begin{aligned} \partial_t^2 K * u\chi_H &= \partial_t K * (\partial_t u)\chi_H + \partial_t K * u\delta_{t=0} && (4) \\ &= K * (\partial_t^2 u)\chi_H + K * (\partial_t u)\delta_{t=0} + \partial_t(K * u\delta_{t=0}). && (5) \end{aligned}$$

(3 Points)

(d) Hence write down a representation formula for u in terms of the given data f, g, h, similar to the others in this course. *(3 Points)*

Solution.

(a) Let us do some preparation calculations.

$$\int_{-t}^{t} \partial_x \partial_x \varphi \, dx = (\partial_x \varphi)(t, t) - (\partial_x \varphi)(-t, t)$$

$$\begin{aligned} \partial_t \left[\int_{-t}^{t} \partial_t \varphi \, dx \right] &= (\partial_t \varphi)(t, t) - (\partial_t \varphi)(-t, t) \cdot (-1) + \int_{-t}^{t} \partial_t^2 \varphi \, dx \\ &= (\partial_t \varphi)(t, t) + (\partial_t \varphi)(-t, t) + \int_{-t}^{t} \partial_t^2 \varphi \, dx. \end{aligned}$$

Note carefully the difference between $(\partial_t \varphi)(-t, t)$ and $\partial_t(\varphi(-t, t))$ which we have discussed several times. I use brackets here, which I would not normal use, to highlight this difference.

$$\begin{aligned} \partial_t(\varphi(t, t)) &= (\partial_x \varphi)(t, t) + (\partial_t \varphi)(t, t) \\ \partial_t(\varphi(-t, t)) &= -(\partial_x \varphi)(-t, t) + (\partial_t \varphi)(-t, t). \end{aligned}$$

With these pieces prepared, we can do the main calculation without too much distraction.

$$\int_0^\infty \int_{-t}^t (\partial_t^2 - \partial_x^2)\varphi(x,t)\,dx\,dt = \int_0^\infty \left[\int_{-t}^t \partial_t^2\varphi(x,t)\,dx - \int_{-t}^t \partial_x^2\varphi(x,t)\,dx\right]\,dt$$
$$= \int_0^\infty \left[\partial_t\left[\int_{-t}^t \partial_t\varphi\,dx\right] - (\partial_t\varphi)(t,t) - (\partial_t\varphi)(-t,t) - (\partial_x\varphi)(t,t) + (\partial_x\varphi)(-t,t)\right]\,dt$$
$$= \int_0^\infty \partial_t\left[\int_{-t}^t \partial_t\varphi\,dx - \varphi(t,t) - \varphi(-t,t)\right]\,dt$$
$$= 0 - \int_{-0}^0 \partial_t\varphi\,dx + \varphi(0,0) + \varphi(-0,0) = 2\varphi(0,0).$$

(b) (1) The delta distribution is the identity (neutral element) of convolution.
(2) $\delta = (\partial_t^2 - \partial_x^2)K$ by part (a), and convolution is a bilinear operator.
(3) The derivative property of the convolution tells us $\partial(f * g) = \partial f * g = f * \partial g$. We use this to move the ∂_x derivatives onto the second operand. χ_H is constant in the x direction, so we just have $\partial_x^2 u$. But u solves the inhomogeneous heat equation, so we can substitute in $\partial_t^2 u - f$.

(c) (4) We move the ∂_t derivative across the convolution operator similar to (3). But χ_H is not constant in the t direction, it has a jump singularity. Therefore we should compute its derivative in the sense of distributions:

$$\partial_t\chi_H(\varphi) = -\chi_H(\partial_t\varphi) = -\int_{\mathbb{R}}\int_0^\infty \partial_t\varphi\,dt\,dx = \int_{\mathbb{R}} \varphi(x,0)\,dx$$

This leads us to define $\delta_{t=0}$ as the distribution $\varphi \mapsto \int_{\mathbb{R}} \varphi(x,0)\,dx$.
(5) The same action as (4), moving the ∂_t across the convolution, is performed again on the first term. In the last term, we take the derivative outside the convolution (this is needed for part (d)).

(d) Putting the calculations of parts (b) and (c) together gives

$$u(x,t) = K * (\partial_t u)\delta_{t=0} + \partial_t(K * u\delta_{t=0}) + K * f\chi_H.$$

We handle the terms one at a time. The third term is the easiest because everything is a function

$$K * f\chi_H(x,t) = \int_{\mathbb{R}^2} K(x-y, t-s) f(y,s)\chi_{[0,\infty)}(s)\,dy\,ds = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds$$

The other two terms have distributions which do not correspond to functions, so we really need to think of them as distributions. When they act on a test function φ we have to unroll the definition of convolution of a distribution $(\partial_t u)\delta_{t=0}$ and a function K, and the definition of the multiplication of a function and a distribution in $(\partial_t u)\delta_{t=0}$. Both of these definitions can be found in Section 2.4 of the script.

$$\begin{aligned} K * (\partial_t u)\delta_{t=0}(\varphi) &= (\partial_t u)\delta_{t=0}(\varphi * \mathrm{P}K) = \delta_{t=0}((\partial_t u)(\varphi * \mathrm{P}K)) \\ &= \int_{\mathbb{R}} \partial_t u(x,0)\,(\varphi * \mathrm{P}K)(x,0)\;dx \\ &= \int_{\mathbb{R}} \partial_t u(x,0) \int_{\mathbb{R}^2} \mathrm{P}K(x-y, 0-s)\varphi(y,s)\;dy\,ds\;dx \\ &= \int_{\mathbb{R}} \partial_t u(x,0) \int_{\mathbb{R}^2} K(y-x, s)\varphi(y,s)\;dy\,ds\;dx \\ &= \int_{\mathbb{R}^2} \varphi(y,s) \left(\int_{\mathbb{R}} \partial_t u(x,0) K(y-x,s)\;dx \right)\;dy\,ds. \end{aligned}$$

What this shows is that the distribution $K * (\partial_t u)\delta_{t=0}$ actually comes from the function

$$(y,s) \mapsto \int_{\mathbb{R}} \partial_t u(x,0) K(y-x,s)\;dx = \chi_{[0,\infty)}(s)\frac{1}{2}\int_{y-s}^{y+s} h(x)\;dx.$$

For the final term, the middle one, we now see why we wrote it in the form $\partial_t(K * u\delta_{t=0})$: because we just calculated something similar, so we can immediately say that the distribution inside the bracket corresponds to

$$(x,t) \mapsto \chi_{[0,\infty)}(t)\frac{1}{2}\int_{x-t}^{x+t} u(z,0)\;dz = \chi_{[0,\infty)}(t)\frac{1}{2}\int_{x-t}^{x+t} g(z)\;dz.$$

We now differentiate this with respect to t, for $t > 0$, to get

$$\frac{1}{2}\left[g(x+t) - g(x-t)\cdot(-1)\right] = \frac{1}{2}\left[g(x+t) + g(x-t)\right].$$

Putting these three terms together, we get the representation formula

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(z)\;dz + \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\;dy\,ds$$

for $t > 0$. We see that this is D'Alembert's formula, with the extra term dealing with the inhomogeneity coming from Duhamel's principle. I hope that this shows you that the representation formulas in this course are not falling from heaven, there is a deeper and unified theory behind them based on distributions and fundamental solutions.