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Introduction to Partial Differential Equations
Exercise sheet 13

29th November, 2021

41. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = 0 \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = 1$ and $h(x) = \cos x$. *(3+2 Points)*

42. Waves and Distributions.

Recall from Analysis the definition of the directional derivative D_v in the direction $v \in \mathbb{R}^n$ is given by

$$D_v f = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}.$$

The partial derivatives are special cases of the directional derivative, and for a function that is differentiable we know that $D_v f = v \cdot \nabla f$.

(a) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $(x,t) \mapsto F(x+t)$ is differentiable with respect to $D_{(1,-1)}$. In fact, show that this directional derivative is zero. (Note, F might not be differentiable and this doesn't matter.) *(2 Points)*

(b) Explain why, for smooth functions, $\partial_t^2 - \partial_x^2 = D_{(1,-1)} D_{(1,1)}$. *(2 Points)*

(c) Consider the distribution $\tilde{F}$ given by $\tilde{F}(\varphi) = \int_{\mathbb{R}^2} F(x+t)\varphi(x,t)\ dx\, dt$. (Optional: show this is a distribution.) Show that it solves the wave equation $(\partial_t^2 - \partial_x^2)\tilde{F} = 0$ in the sense of distributions. *(2 Points)*

One can apply the same reasoning to show that $F(x-t)$ also solves the wave equation in the weak sense. It is for this reason we can claim that the general solution to the wave equation in one-dimension is $F(x+t) + G(x-t)$ for any continuous functions F and G. To extend this result to all distributions, it's best to proceed with the translation operator from Sheet 6 Question 18(g) and those sort of methods.

43. Represent.

In this question we derive a representation formula for the one dimensional wave equation, similar to the Poisson representation formula and Corollary 4.5. The difference is that we will try to do as much of the proof as we can using distributions and convolution algebra and avoiding explicit integrals.

We have not proved all the results needed to give a rigorous proof, but I think they are believable. Namely, you are free to assume the following two facts:
(i) the Leibniz rule holds for the derivative of a product of a smooth function and a distribution $\partial_i(gF) = (\partial_i g)F + g(\partial_i F)$, and

(ii) the derivative property of convolutions also holds for distributions $\partial_i(g * F) = (\partial_i g) * F = g * (\partial_i F)$.

Let χ_A be the characteristic function for the set A. That means $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise.

(a) Let $K : \mathbb{R}^2 \to \mathbb{R}$ be defined as $K(x,t) = \frac{1}{2}\chi_{\{t\geq 0\}}\chi_{\{-t\leq x\leq t\}}$. Show that $(\partial_t^2 - \partial_x^2)K = \delta$ in the sense of distributions. That is, show for all test functions $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ that

$$\int_0^\infty \int_{-t}^{t} \frac{1}{2}(\partial_t^2 - \partial_x^2)\varphi(x,t)\, dx\, dt = \varphi(0,0).$$

This shows that K is a fundamental solution of the wave equation. *(3 Points)*

(b) Suppose that $u : \mathbb{R} \times [0,\infty)$ is a (smooth) solution to the inhomogeneous wave equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x). \end{cases}$$

We can extend this to the whole plane by considering $u\chi_H$ for $H := \mathbb{R} \times [0,\infty)$ and likewise for the functions f, g, h. Explain and/or add further steps of working to the following calculation: For $t > 0$

$$\begin{aligned} u(x,t) &= \delta * u\chi_H && (1) \\ &= \partial_t^2 K * u\chi_H - \partial_x^2 K * u\chi_H && (2) \\ &= \partial_t^2 K * u\chi_H - K * (\partial_t^2 u)\chi_H + K * f\chi_H. && (3) \end{aligned}$$

(5 Points)

(c) Explain and/or add further steps of working to the following calculation, including the correct definition of $\delta_{t=0}$.

$$\begin{aligned} \partial_t^2 K * u\chi_H &= \partial_t K * (\partial_t u)\chi_H + \partial_t K * u\delta_{t=0} && (4) \\ &= K * (\partial_t^2 u)\chi_H + K * (\partial_t u)\delta_{t=0} + \partial_t(K * u\delta_{t=0}). && (5) \end{aligned}$$

(3 Points)

(d) Hence write down a representation formula for u in terms of the given data f, g, h, similar to the others in this course. *(3 Points)*