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Introduction to Partial Differential Equations
Exercise sheet 11

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38. One step at a time.

Prove the following identity for the heat kernel in one dimension ($n = 1$):

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s)\, dy.$$

Interpret this equation in the context of the heat equation on the line. *(4 Points)*

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$

Solution.

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s)\, dy &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}\sqrt{4\pi s}} \exp\left(-\frac{(x-y)^2}{4t} - \frac{y^2}{4s}\right) dy \\
&= \frac{1}{\sqrt{4\pi t}\sqrt{4\pi s}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{4t} + \frac{x}{2t}y - \left[\frac{1}{4t} + \frac{1}{4s}\right] y^2\right) dy \\
&= \frac{1}{\sqrt{4\pi t}\sqrt{4\pi s}} \sqrt{\frac{\pi}{\frac{s+t}{4st}}} \exp\left(\frac{x^2}{4t^2} \cdot \frac{1}{4} \cdot \frac{4st}{s+t} - \frac{x^2}{4t}\right) \\
&= \frac{1}{\sqrt{4\pi(s+t)}} \exp\left(-\frac{x^2}{4(s+t)}\right) = \Phi(x, s+t).
\end{aligned}$$

We know that convolution of the initial condition with the fundamental solution over the space coordinates gives the solution to the initial value problem. The fact that we have $s + t$ in the time suggests we should do this twice. So if we begin with the initial condition $h(x)$ and then solve up to time t, we have

$$u(x, t) = \int_{\mathbb{R}} \Phi(x-y, t)h(y)\, dy.$$

If we take $x \mapsto u(x, t)$ as the start of a new initial value problem and then calculate what the solution is at time s we get

$$\begin{aligned}
v(x, s) &= \int_{\mathbb{R}} \Phi(x-y, s)u(y, t)\, dy \\
&= \int_{\mathbb{R}} \Phi(x-y, s) \int_{\mathbb{R}} \Phi(y-z, t)h(z)\, dz\, dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi(x-y, s)\Phi(y-z, t)\, dy\right) h(z)\, dz \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi(x-z-y, s)\Phi(y, t)\, dy\right) h(z)\, dz.
\end{aligned}$$

(We made a shift substitution to get to the last line, but reused the letter y.) On the other hand, if we began with the initial condition $h(x)$ and computed the solution at time $s + t$ we would find

$$u(x, s+t) = \int_{\mathbb{R}} \Phi(x-z, s+t)h(z)\, dz.$$

The relation we just proved says that these two solutions are the same. This shows that the heat equation does not have 'long term memory', it only depends on the immediately prior state, not on anything that happened before that. This is in contrast to its behaviour in space, where the action at one point can immediately affect points infinitely far away. This property is also called the semigroup property because the solution operators $h \mapsto H_t h := \Phi(\cdot, t) * h$ form a semigroup: $H_s H_t = H_{s+t}$.

39. Some like it hot.

Find the solution $u : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem using the heat kernel of $[0,1]$:

$$\begin{cases} \dot{u} - 3\partial_{xx} u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = \sin(\pi x) + 2\sin(5\pi x) & \text{for } x \in (0,1). \end{cases}$$

(6 Points)

Solution. Firstly, this isn't quite the heat equation, but we can rescale time to absorb the factor 3, namely $s = 3t$. We use the heat kernel for the interval $[0,1]$. By Theorem 4.16

$$\begin{aligned} u(x,s) &= 0 - 0 + \int_{[0,1]} [\sin(\pi x) + 2\sin(5\pi x)]\, H_{[0,1]}(x,y,s)\ \mathrm{d}y \\ &= \int_0^1 [\sin(\pi x) + 2\sin(5\pi x)] \sum_{k=1}^{\infty} e^{-\pi^2 k^2 s} 2\sin(k\pi x)\sin(k\pi y)\ \mathrm{d}y \\ &= 2\sum_{k=1}^{\infty} e^{-\pi^2 k^2 s} \sin(k\pi x) \left[\int_0^1 \sin(\pi y)\sin(k\pi y)\ \mathrm{d}y + 2\int_0^1 \sin(5\pi y)\sin(k\pi y)\ \mathrm{d}y\right]. \end{aligned}$$

Either by applying integration by parts twice or quoting the result on pg 72 of the script, we see that $\int_0^1 \sin(m\pi z)\sin(n\pi z)\, dz$ is zero if $m \neq n$ and is 0.5 if they are equal. Thus all but two terms of the sum are zero, namely $k = 1, 5$.

$$\begin{aligned} u(x,s) &= 2e^{-\pi^2 s}\sin(\pi x)\, 0.5 + 2e^{-25\pi^2 s}\sin(5\pi x)\, 2 \cdot 0.5 \\ &= e^{-\pi^2 s}\sin(\pi x) + 2e^{-25\pi^2 s}\sin(5\pi x) \\ u(x,t) &= e^{-3\pi^2 t}\sin(\pi x) + 2e^{-75\pi^2 t}\sin(5\pi x) \end{aligned}$$

The temperature falls very quickly, so make sure you have a jacket. We can also check our solution:

$$\begin{aligned} \dot{u}(x,t) &= -3\pi^2 e^{-3\pi^2 t}\sin(\pi x) - 150\pi^2 e^{-75\pi^2 t}\sin(5\pi x) \\ \partial_{xx} u(x,t) &= -\pi^2 e^{-3\pi^2 t}\sin(\pi x) - 50\pi^2 e^{-75\pi^2 t}\sin(5\pi x). \end{aligned}$$

Because this is just the 1-dimensional heat equation, there are a variety of other effective methods: separation of variables to reduce it to two ODES, or a Laplace transform to reduce it to an inhomogeneous ODE are two that spring to mind.

Another approach to the above integrals would be to write the heat kernel in the form $\hat{\Phi}(n,t)e^{2\pi inx}$. Then again all but two of the integrals will be zero and we see the solution is the sum of two Fourier transforms of the fundamental solution.

40. The heat kernel on $\mathbb{S}^1$.

(Adapted from Exercise 4.22 in the lecture script) Denote the fundamental solution of the heat equation by $\Phi(x,t)$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Show that

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

defines a smooth periodic function with period 1 (i.e. $\tilde{f}(x+1) = \tilde{f}(x)$). *(2 Points)*

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, with period 1, and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ a solution to the heat equation with initial condition $u(x,0) = h(x)$. Show that u remains periodic in the spatial coordinate for all time. *(2 Points)*

(c) Conclude that

$$u(x,t) := \int_{\mathbb{S}^1} h(y) \sum_{n\in\mathbb{Z}} \Phi(x-y+n,t).$$

solves the heat equation with the initial condition. *(2 Points)*

(d) Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}.$$

Show, with the aid of this equality, the relation

$$\Theta(x-y, 4\pi it) = \sum_{n\in\mathbb{Z}} \Phi(x-y+n,t),$$

where the left hand side is the Jacobi's theta function from Section 4.7.

(2 Points)

(e) How would you modify Definition 4.14 to give give an abstract definition of the heat kernel $H_{\mathbb{S}^1}$? *(2 Points)*

Solution.

(a) The function is period with period 1 by renaming the summation variable. The difficulty is proving that it converges. But this follows by comparison to the series $\frac{1}{n^2}$. We know that there is a constant C with $|f| < \frac{C}{x^2}$ for all x. Then for $x \in [0,1]$ we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} |f(x+n)| &\leq |f(x)| + |f(x-1)| + C \sum_{n\neq 0,-1} \frac{1}{(x+n)^2} \\ &\leq |f(x)| + |f(x-1)| + C\sum_{n\neq 0} \frac{1}{n^2} \\ &\leq 2\sup|f| + C\frac{\pi^2}{6},\end{aligned}$$

which demonstrates uniform convergence. The derivative of a Schwartz function is a Schwartz function, so the same result applies to the sum of the derivatives and shows $\tilde{f}$ is differentiable. Repeating the argument with $\partial_i \tilde{f}$ proves $\tilde{f}$ is smooth.

(b) Consider the difference of $v(x,t) = u(x+1,t) - u(x,t)$. This solves the heat equation with the initial condition $v(x,0) = h(x+1) - h(x) = 0$. Then we use Theorem 4.12 for this PDE to conclude that v must be zero.

(c) The initial condition of the heat equation on the circle is given by $h : [0,1] \to \mathbb{R}$ with $h(0) = h(1)$. Thus we can extend h to a period-1 function on all of $\mathbb{R}$.

By the previous questions, we know that $\tilde{\Phi}$ is a well defined smooth and periodic function and that we can pass an integral through the summation:

$$\begin{aligned} \int_{\mathbb{S}^1} h(y) \sum_{n\in\mathbb{Z}} \Phi(x-y+n,t)\, dy &= \sum_{n\in\mathbb{Z}} \int_0^1 h(y)\Phi(x-(y-n),t)\, dy \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} h(z+n)\Phi(x-z,t)\, dz \\ &= \int_{\mathbb{R}} h(z)\Phi(x-z,t)\, dz. \end{aligned}$$

We know from Theorem 4.3 that this solves the heat equation.

(d) One could try to make an argument using a uniqueness of the heat kernel but we will proceed with the hint suggested. Using the previous parts of this question

$$\begin{aligned} \tilde{\Phi}(z,t) &= \sum \hat{\Phi}(n,t) e^{2\pi i n z} \\ &= \sum \exp(-4t\pi^2 n^2) e^{2\pi i n z} \\ &= \sum \exp(2\pi i n z + \pi i (4\pi i t) n^2) \\ &= \Theta(z, 4\pi i t) \end{aligned}$$

(e) Let's read Definition 4.14 and see what does and does not apply on the circle. We can't use the idea $\mathbb{S}^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$, because it is not open. And otherwise $\mathbb{S}^1$ is not a subset of $\mathbb{R}^n$. But we can overcome this by using periodic functions on $\mathbb{R}$. "The heat kernel $H_{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^+ \to \mathbb{R}$ is a function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ that is periodic in the first two variables with periods 1."

Condition (i) is trivially true because the circle is already closed. So we can omit this condition.

For condition (ii), we have the problem that Φ is not periodic. However we can replace this with $\tilde{\Phi}$ which is periodic. So $(y,t) \mapsto H_{\mathbb{S}^1}(x,y,t) - \tilde{\Phi}(x-y,t)$ should solve the homogeneous heat equation and extend continuously to 0 on $(y,t) \in \mathbb{S}^1 \times \{0\}$.

Note then that $H_{\mathbb{S}^1}(x,y,t) = \tilde{\Phi}(x-y,t)$ itself meets this definition. This is exactly parallel to the way that $\Phi(x-y)$ is the Green's function of the Laplacian on $\mathbb{R}^n$ and $\Phi(x-y,t)$ is the heat kernel on $\mathbb{R}^n$.