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Introduction to Partial Differential Equations
Exercise sheet 11

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38. One step at a time.

Prove the following identity for the heat kernel in one dimension ($n = 1$):

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s)\, dy.$$

Interpret this equation in the context of the heat equation on the line. *(4 Points)*

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$

39. Some like it hot.

Find the solution $u : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem using the heat kernel of $[0,1]$:

$$\begin{cases} \dot{u} - 3\partial_{xx}u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = \sin(\pi x) + 2\sin(5\pi x) & \text{for } x \in (0,1). \end{cases}$$

(6 Points)

40. The heat kernel on $\mathbb{S}^1$.

(Adapted from Exercise 4.22 in the lecture script) Denote the fundamental solution of the heat equation by $\Phi(x,t)$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Show that

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

defines a smooth periodic function with period 1 (i.e. $\tilde{f}(x+1) = \tilde{f}(x)$). *(2 Points)*

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, with period 1, and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ a solution to the heat equation with initial condition $u(x,0) = h(x)$. Show that u remains periodic in the spatial coordinate for all time. *(2 Points)*

(c) Conclude that

$$u(x,t) := \int_{\mathbb{S}^1} h(y) \sum_{n\in\mathbb{Z}} \Phi(x-y+n, t).$$

solves the heat equation with the initial condition. *(2 Points)*

(d) Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}.$$

Show, with the aid of this equality, the relation

$$\Theta(x-y, 4\pi it) = \sum_{n\in\mathbb{Z}} \Phi(x-y+n, t),$$

where the left hand side is the Jacobi's theta function from Section 4.7.

(2 Points)

(e) How would you modify Definition 4.14 to give give an abstract definition of the heat kernel $H_{\mathbb{S}^1}$? *(2 Points)*