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**Introduction to Partial Differential Equations**
**Exercise sheet 11**

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**35. Sugar, we're going down swinging.**

First let $\Omega' \subset \mathbb{R}^n \times \mathbb{R}$ be an open and connected region. A function $v : \Omega' \to \mathbb{R}$ is called a *sub-solution* of the heat equation if $\dot{v} - \Delta v \leq 0$.

**(a)** *Mean value estimate for sub-solutions* Take any point $(x,t) \in \Omega'$ and a small radius $r > 0$ so that $E(x,t,r) \in \Omega'$ (refer to Definition 4.6). Modify the proof the mean value property of the heat equation to show that

$$v(x,t) \leq \frac{1}{4r^n} \int_{E(x,t,r)} v(y,s) \frac{|x-y|^2}{|t-s|^2} \, d^n y \, ds$$

holds for all sub-solutions. *(2 Points)*

Now let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and path connected region. We denote the parabolic cylinder of $\Omega$ by $\Omega_T := \Omega \times (0,T]$ as in Section 4.4. Suppose that $v : \Omega_T \to \mathbb{R}$ is a sub-solution that extends continuously to $\overline{\Omega_T}$.

**(b)** *Maximum principle for sub-solutions* Following on from (a), establish that if $v$ takes the value $\sup_{\Omega_T} v$ on $\Omega_T$, then it is constant. *(2 Points)*

**(c)** *A monotonicity property* For $j \in \{1,2\}$ let $f_j : \Omega \times (0,T) \to \mathbb{R}$, $h_j : \Omega \to \mathbb{R}$, and $g_j : \partial\Omega \times [0,T]$ be smooth functions, and likewise let $u_j : \Omega \times (0,T)$ be smooth functions with continuous extensions to the boundary that satisfy

$$\begin{cases} \dot{u}_j - \Delta u_j = f_j & \text{on } \Omega \times (0,T) \\ u_j(x,0) = h_j(x) & \text{on } \Omega \\ u_j = g_j & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

Suppose further that $f_1 \leq f_2$, $g_1 \leq g_2$, and $h_1 \leq h_2$. Show in this case that $u_1 \leq u_2$ as well. *(2 Points)*

**36. Heat death of the universe.**

First a corollary to Theorem 4.3:

**(a)** Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u$ is defined as in Theorem 4.3. Show

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}} \|h\|_{L^1}.$$

*(2 Points)*

The above corollary shows how solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ with such initial conditions behave: they tend to zero as $t \to \infty$. Physically this is because if $h \in L^1$ then there is a finite amount of total heat, which over time becomes evenly spread across the plane.

On open and bounded domains $\Omega \subset \mathbb{R}^n$ we can have different behaviour, due to the boundary conditions holding the temperature steady. In this question we determine the long time behaviour of solutions $u$ to the heat equation on open and bounded sets $\Omega$ with $u(x,t) = g(x)$ on $\partial\Omega \times \mathbb{R}^+$ and $u(x,0) = h(x)$. Assume that there is a steady state solution, i.e. a solution to the Dirichlet problem for the Laplace equation $\Delta v = 0$ and $v|_{\partial\Omega} = g$. We claim $u \to v$ as $t \to \infty$. Let $w(x,t) = u(x,t) - v(x)$. The claim is equivalent to $w \to 0$.

**(b)** What PDE and boundary conditions does $w$ obey? *(2 Points)*

**(c)** Let $l_m$ be the function from Theorem 4.3 that solves heat equation on $\mathbb{R}^n$ with $l_m(x,0) = mk(x)$ for $m$ a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that $k|_\Omega \equiv 1$. Why must $k$ exist? Why does $l_m \to 0$ as $t \to \infty$? What boundary conditions on $\Omega$ does it obey? *(3 Points)*

**(d)** Use the monotonicity property to show that $w$ tends to zero. *(2 Points)*
Hint. Consider $a = \sup_{x\in\Omega} |w(x,0)|$.

**37. The Fourier transform.**

Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined in Section 4.6 to be a function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y} h(y)\, dy.$$

Lemma 4.20 shows that it is well-defined for Schwartz functions.

**(a)** Give the definition of a Schwartz function. *(1 Point)*

**(b)** Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function. *(1 Point)*

**(c)** Show that the Fourier transform of $\exp(-A^2x^2)$ for a constant $A > 0$ is $\sqrt{\pi}A^{-1}\exp(-\pi^2k^2A^{-2})$. You may use that $\int_{\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$. *(2 Points)*

**(d)** Show that $\widehat{\partial_j f}(k) = 2\pi i k_j \hat{f}(k)$ for Schwartz functions $f : \mathbb{R}^n \to \mathbb{R}$. *(2 Points)*

**(e)** If $u : \mathbb{R} \times \mathbb{R}^+$ is a solution to the heat equation, we can apply a Fourier transform in the space coordinate to get a function $\hat{u}(k,t)$. Show that this function obeys

$$\frac{\partial \hat{u}}{\partial t} + 4\pi^2k^2\hat{u} = 0.$$

Solve this ODE in the time variable. *(2 Points)*

**(f)** Suppose that we have the initial condition $u(x,0) = h(x)$ for $x \in \mathbb{R}$ for a Schwartz function $h$. Then $\hat{u}(k,0) = \hat{h}(k)$. Apply the inverse Fourier transformation to rederive the solution given in Theorem 4.3. *(2 Points)*