Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 10

8th November, 2021

32. Special solutions of the heat equation.

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 Point)*

(b) Look for "separable" solutions of the heat equation: those of the form $u(x,t) = X(x)T(t)$. Argue that there is constant λ such that

$$\dot{T}(t) = -\lambda T(t), \qquad -\Delta X(x) = \lambda X(x),$$

for all x and t. *(2 Points)*

(c) Suppose that Ω is a bounded domain, and that $u|_{\partial\Omega} = 0$. Apply the Green's first formula with $v = u$ to an eigenfunction of Δ to show that λ can only be positive. *(2 Points)*

(d) How do separable solutions behave over time? *(1 Point)*

Solution.

(a) Steady-state solutions are defined by $\dot{u} = 0$. The heat equation then reduces to a Poisson equation: $0 - \Delta u = f$.

(b) Substitution and rearrangement gives

$$\frac{\dot{T}(t)}{T(t)} = \frac{\Delta X(x)}{X(x)}.$$

Fix a value of x and let t vary. The right hand side does not depend on t, so remains constant. We call this constant $-\lambda$. Thus we get $\frac{\dot{T}(t)}{T(t)} = -\lambda$. Likewise the right hand side must also equal $-\lambda$.

(c) On one hand, because u is an eigenfunction with $-\Delta u = \lambda u$ we have

$$\int_\Omega u\Delta u \; dx = -\lambda \int_\Omega |u|^2 \; dx.$$

On the other hand, using Green's first formula

$$\int_\Omega u\Delta u \; dx = -\int_\Omega \nabla u \cdot \nabla u \; dx + \int_{\partial\Omega} u\nabla u \cdot N \; d\sigma = -\int_\Omega \|\nabla u\|^2 \; dx.$$

From this we see that $\lambda \geq 0$. But the only solution for $\lambda = 0$ is $u \equiv 0$. Hence 0 is not an eigenvalue and $\lambda > 0$.

(d) If u is a separable solution, then X is an eigenfunction and so λ is an eigenvalue. We have seen in part(c) that λ is always a positive number. The part of u that changes with time is $T(t)$. This obeys the ODE $\dot{T} = -\lambda T$ and so $T(t) = T_0 \exp(-\lambda t)$. Therefore separable solutions decay exponentially. (This is also the reason we have an 'extra' negative sign).

33. Geothermal Power.

Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x,0) = h(x)$. Suppose, as an ansatz, that the solution is a power series in t, i.e. $u(x,t) = \sum_{k=0}^{\infty} a_k(x)\, t^k$ for functions $a_k : \mathbb{R}^n \to \mathbb{R}$.

(a) Why is $a_0 = h$? *(1 Point)*

(b) Show that the a_k obey the recursion relation $a_{k+1} = \frac{1}{k+1}\Delta a_k$. *(2 Points)*

(c) Hence conclude that $u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}(\Delta^k h)(x) t^k$. *(1 Point)*

(d) Suggest some conditions on h that would ensure this series converges. *(1 Bonus Point)*

Solution.

(a) Apply the initial condition. All terms except $k = 0$ vanish.

(b) We differentiate

$$\dot{u} = \sum_{k=1}^{\infty} k\, a_k\, t^{k-1} = \sum_{k=0}^{\infty} (k+1)\, a_{k+1}\, t^k, \qquad \Delta u = \sum_{k=0}^{\infty} (\Delta a_k) t^k.$$

Equating the coefficients gives the recursion relation.

(c) We get

$$a_k = \frac{1}{k}\Delta a_{k-1} = \frac{1}{k}\Delta\left(\frac{1}{k-1}\Delta a_{k-2}\right) = \frac{1}{k(k-1)}\Delta^2 a_{k-2} = \cdots = \frac{1}{k!}\Delta^k a_0. = \frac{1}{k!}\Delta^k h.$$

(d) We have a power series whose coefficients belong to the set of smooth functions. If the coefficients were to belong to a Banach space with norm $\|\cdot\|$, then the series converges for all times when $\limsup \sqrt[k]{\|a_k\|} = 0$. An example of such Banach space would be if h was a function of compact support with the supremum norm.

More directly, suppose that there is a constant $M > 0$ with

$$|(\Delta^k f)(x)| \leq M^k,$$

for all x and $k \geq 0$. This is sufficient to ensure convergence for all time.

34. The distribution of heat.

Consider the fundamental solution of the heat equation $\Phi(x,t)$ given in Definition 4.1.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 Points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 Points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x,t)\varphi(x,t)\; dx\, dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)\ dx\, dt\ \ ?$$

(1 Point)

(d) Conclude with the help of Lemma 4.2 and Theorem 4.3 that

$$|H(\varphi)| \le T\, \|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(3 Points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.3 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\ dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 Point)*

(f) Hence show that

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\ dy\, dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(4 Points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\ dy\, dt \to 0$$

(2 Points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^\varepsilon + \int_\varepsilon^\infty\right) \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\ dy\, dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.

Solution.

(a) For $t > 0$, $\Phi(x,t)$ and all its derivatives have the form $t^{-k}q(x,t)\exp(-x^2/4t)$ for $k \in \mathbb{N}_0$ and q a polynomial. As $t \to 0^+$ for $x \neq 0$, the exponential terms is dominant and forces the expression to zero. For $t < 0$ the heat kernel is identically zero, and so all its derivatives are zero and the limits as $t \to 0^-$ is zero. Thus we have the smooth extension $\Phi(x,0) = 0$ for $x \neq 0$.

(b) By direct calculation, for $t > 0$

$$\begin{aligned}
(4\pi)^{n/2}\partial_t\Phi &= -\frac{n}{2}t^{-n/2-1}e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{4}t^{-n/2-2}e^{-\frac{|x|^2}{4t}} \\
(4\pi)^{n/2}\partial_j\Phi &= -\frac{x_j}{2}t^{-n/2-1}e^{-\frac{|x|^2}{4t}} \\
(4\pi)^{n/2}\partial_j^2\Phi &= -\frac{1}{2}t^{-n/2-1}e^{-\frac{|x|^2}{4t}} + \frac{x_j^2}{4}t^{-n/2-2}e^{-\frac{|x|^2}{4t}}.
\end{aligned}$$

The appropriate sum gives zero. We see that all derivatives of the function are zero for $t = 0, x \neq 0$. Therefore the heat equation holds there too.

(c) Because φ has compact support, it is zero outside $B(0,R) \times [-T,T]$ for some positive constants R and T. Additionally, we know that Φ is zero for $t < 0$. Therefore the integrand is zero outside $B(0,R) \times [0,T]$ and can be discarded.

(d) First just apply the estimate that bounds φ:

$$|H(\varphi)| \leq \int_0^T \int_{\mathbb{R}^n} \Phi(x,t) \|\varphi(x,t)\|_{K,0} \, dx \, dt = \|\varphi(x,t)\|_{K,0} \int_0^T \int_{\mathbb{R}^n} \Phi(x,t) \, dx \, dt.$$

So it remains to bound the integral of Φ over this region. Consider the function $g(t) := \int_{\mathbb{R}^n} \Phi(x,t) \, dx$. Lemma 4.2 says that $g(t) = 1$ for $t > 0$. Theorem 4.3(iii) says that $g(0) = 1$. This gives

$$\int_0^T \int_{\mathbb{R}^n} \Phi(x,t) \, dx \, dt = \int_0^T 1 \, dt = T$$

as the constant.

(e) One only needs to modify one step in the proof of Theorem 4.3: $|h(y,s) - h(x,s)|$ is bounded by twice the supremum of h over space *and time* variables.

(f) We should try to apply integration by parts, in order to move the derivatives from φ to Φ, because we know what Φ is. However the boundary term does not necessarily vanish on the $t = \varepsilon$ plane.

$$\begin{aligned} -\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi \partial_t \varphi - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi \Delta \varphi &= -\left[\int_{\mathbb{R}^n} \Phi \varphi \Big|_{t=\varepsilon}^{t=\infty} - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \partial_t \Phi \varphi \right] - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Delta \Phi \varphi \\ &= \int_{\mathbb{R}^n} \Phi(x,\varepsilon) \varphi(x,\varepsilon) + \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\partial_t \Phi - \Delta \Phi) \varphi \\ &= \int_{\mathbb{R}^n} \Phi(x,\varepsilon) \varphi(x,\varepsilon) \\ &= \int_{\mathbb{R}^n} \Phi(0 - x, \varepsilon) \varphi(x,\varepsilon). \end{aligned}$$

The second integral on the second line vanishes due to part (b). We can now apply the previous part to conclude that this limits to $\varphi(0,0)$. Notice the need for uniform convergence, because the second parameter of φ is also being changed by the limit $\varepsilon \to 0$.

(g) We can estimate the norm of h out

$$\left| \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(x,t) h(x,t) \, dx \, dt \right| \leq \|h\|_\infty \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(x,t) \, dx \, dt. = \|h\|_\infty \int_0^\varepsilon dt = \|h\|_\infty \varepsilon.$$

Clearly this tends to zero.