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Introduction to Partial Differential Equations
Exercise sheet 10

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32. Special solutions of the heat equation.

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 Point)*

(b) Look for "separable" solutions of the heat equation: those of the form $u(x,t) = X(x)T(t)$. Argue that there is constant λ such that

$$\dot{T}(t) = -\lambda T(t), \qquad -\Delta X(x) = \lambda X(x),$$

for all x and t. *(2 Points)*

(c) Suppose that Ω is a bounded domain, and that $u|_{\partial\Omega} = 0$. Apply the Green's first formula with $v = u$ to an eigenfunction of Δ to show that λ can only be positive. *(2 Points)*

(d) How do separable solutions behave over time? *(1 Point)*

33. Geothermal Power.

Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x,0) = h(x)$. Suppose, as an ansatz, that the solution is a power series in t, i.e. $u(x,t) = \sum_{k=0}^{\infty} a_k(x)\, t^k$ for functions $a_k : \mathbb{R}^n \to \mathbb{R}$.

(a) Why is $a_0 = h$? *(1 Point)*

(b) Show that the a_k obey the recursion relation $a_{k+1} = \frac{1}{k+1}\Delta a_k$. *(2 Points)*

(c) Hence conclude that $u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}(\Delta^k h)(x) t^k$. *(1 Point)*

(d) Suggest some conditions on h that would ensure this series converges. *(1 Bonus Point)*

34. The distribution of heat.

Consider the fundamental solution of the heat equation $\Phi(x,t)$ given in Definition 4.1.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 Points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 Points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x,t)\varphi(x,t)\; dx\, dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)\; dx\, dt \ ?$$

(1 Point)

(d) Conclude with the help of Lemma 4.2 and Theorem 4.3 that

$$|H(\varphi)| \leq T \, \|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(3 Points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.3 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\;dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 Point)*

(f) Hence show that

$$\int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\;dy\,dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(4 Points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_{0}^{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\;dy\,dt \to 0$$

(2 Points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{\infty} \right) \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\;dy\,dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.