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Introduction to Partial Differential Equations
Exercise sheet 9

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28. Do nothing by halves.

Let $H_1^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \dots, x_n)$ reflection in the plane H^0. Let $Q_{12} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0, x_2 > 0\}$ be a quadrant.

(a) A reflection principle for harmonic functions. Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0. Show that the function $v : \mathbb{R}^n \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic. *(4 Points)*

(b) Green's function for the half-space. Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

(3 Points)

(c) Green's function for the quadrant. Compute the Green's function for Q_{12}. *(3 Points)*

29. Teach a man to fish...

(a) Using the Green's function of H_1^+ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem $\Delta u = 0$ in $H_1^+, u|_{H_1^0} = g$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x - z|^n} \, \mathrm{d}\sigma(z)$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined. *(3 Points)*

(b) Show that if g is periodic (that is, there is some vector $L \in \mathbb{R}^{n-1}$ with $g(x + L) = x$ for all $x \in \mathbb{R}^{n-1}$) then so is the solution. *(2 Points)*

(c) Now consider the plane $n = 2$ with g function with compact support. Approximate the value of $u(x)$ for large $|x|$. Feel free to modify this question as you see fit, what interesting things can you say about the growth of u?

(2 Points + Bonus Points as deserved)

30. One of a kind.

Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected domain and $u, v \in C^2(\overline{\Omega})$ two harmonic functions. Suppose that there is an open subset $U \subset \Omega$ such that $u = v$ on U. Prove that $u = v$ on Ω using Corollary 3.22 (Harmonic functions are analytic). This is called the *unique continuation property* of harmonic functions. *(2 Points)*

31. To be or not to be.

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

(a) Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.24. *(2 Points)*

(b) Generalise this: What are the necessary and sufficient conditions on g for the Dirichlet problem to have a solution on this domain? *(2 Points)*

(c) Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points? *(1 Bonus Point)*