Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 8

25th October, 2021

24. Back in the saddle.

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(2 Points)*

25. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean: $v(x) \leq \Phi(v,x,r)$.

(a) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 Points)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω. *(3 Points)*

(c) Let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that $\|\nabla u\|^2$ is subharmonic. *(2 Points)*

(d) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic. *(1 Point)*

26. Never judge a book by its cover.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(4 Points)*

27. Weak Tea.

Choose any point $x \in \mathbb{R}^n$. For each function $\psi \in C_0^\infty((0,r))$ there is a test function

$$f_{x,\psi}(y) \in C_0^\infty(B(x,r)), \qquad f_{x,\psi(y)} = \begin{cases} \frac{\psi(|y-x|)}{n\omega_n |y-x|^{n-1}} & \text{for } y \neq x \\ 0 & \text{otherwise} \end{cases}$$

given in the definition of the weak mean value property 3.6. We try in this question to develop some intuition for these functions.

(a) Justify that $f_{x,\psi}$ is indeed a test function. *(2 Points)*

(b) For the delta distribution δ_x centred at x, compute $\delta_x(f_{x,\psi})$. *(1 Point)*

(c) Does δ_x satisfy the weak mean value property? Defend your answer. *(2 Points)*

(d) Let λ_ε be a mollifier on $\mathbb{R}$ and define $\psi_\varepsilon(r) = \lambda_\varepsilon(r-1)$. For $\varepsilon < 1$ this obeys $\psi_\varepsilon \in C_0^\infty((1-\varepsilon, 1+\varepsilon)) \subset C_0^\infty((0,2))$. Let $f_\varepsilon = f_{0,\psi_\varepsilon}$ be the corresponding test functions for balls centred at the origin. Let g be a continuous function. Show that as $\varepsilon \to 0$

$$F_g(f_\varepsilon) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{g(y)}{|y|^{n-1}} \, \psi_\varepsilon(|y|) \, dy \to \Phi(g, 0, 1).$$

(Hint. Lemma 2.8.) *(5 Points)*