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Introduction to Partial Differential Equations
Exercise sheet 7

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20. Twirling towards freedom.

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function.

(a) Show that the following functions are also harmonic.

- **(i)** $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$.
- **(ii)** $v(x) = u(ax)$ for $a \in \mathbb{R}$.
- **(iii)** $v(x) = u(Rx)$ for $R(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ the reflection operator.
- **(iv)** $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under all Euclidean motions and harmonic functions can be rescaled. *(5 Points)*

(b) Show, using the chain rule, that in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ the Laplacian is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(3 Points)

(c) Hence show that $v(r,\theta) = u(r^{-1}, \theta)$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$. *(2 Points)*

21. Harmonic Polynomials in Two Variables.

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 Point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x, y) = (x + \iota y)^n$ and let $u_n(x, y)$ and $v_n(x, y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(2 Points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n-1$. *(1 Point)*

(d) Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(3 Bonus Points)*

22. Means and Ends

In the lecture script we often encounter the *spherical mean* of v:

$$\Phi(v, x, r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v(y)\, \mathrm{d}\sigma(y).$$

We have seen in a previous exercise that $\lim_{r\to 0} \Phi(v, x, r) = v(x)$ when v is continuous. Let $v \in C^2(\overline{\Omega})$ be any twice continuously differentiable function. Carefully justify the formula

$$\frac{\partial}{\partial r}\Phi(v, x, r) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta v(x_0 + rz)\, \mathrm{d}z.$$

This formula is used in the proof of the Mean Value property. It shows why spherical means and harmonic functions are related. *(5 Points)*

23. Liouville's Theorem.

Let $u \in C^2(\mathbb{R}^2)$ be a harmonic function. Liouville's theorem (3.5 in the script) says that if u is bounded, then u is constant. In this question we give a geometric proof using *ball means*. Similar to a spherical mean, the ball mean of a function $v \in C(\overline{\Omega})$ is defined when $\overline{B(x,r)} \subset \Omega$:

$$M(v, x, r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y) \, \mathrm{d}y$$

This proof comes from the following article Nelson, 1961.

(a) Show that u obeys the mean value property on balls, $u(x) = M(u, x, r)$.
(Hint. use polar coordinates for the integral $\mathrm{d}y = \mathrm{d}\sigma \, \mathrm{d}\rho$.) *(2 Points)*

(b) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 Bonus Points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(c) Suppose that u is bounded on the plane: $-C \leq u(x) \leq C$ for all x and some constant C. Show that *(2 Points)*

$$\left|M(u,a,r) - M(u,b,r)\right| \leq \frac{2C}{\omega_2} \left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}} \right)$$

(d) Complete the proof that u is constant. *(2 Points)*