Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 6

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17. Distributions.

(a) Choose any compact set $K \subset \mathbb{R}$. Since it is bounded, there exists $R > 0$ with $K \subseteq [-R, R]$. Now choose any test function $\phi \in C_0^\infty(\mathbb{R})$ with compact support in K. Since it is continuous, $\sup_{x\in K} |\phi(x)|$ is finite. Prove the following inequality

$$\left| \int_0^\infty \phi(x)\, dx \right| \leq 2R \sup_{x\in K} |\phi(x)|$$

(2 Points)

(b) Show directly from Definition 2.6 that the Heaviside distribution

$$H : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_0^\infty \phi(x)\, dx$$

is a distribution on $\mathbb{R}$. *(2 Points)*

(c) Calculate and describe the first and second derivatives of the Heaviside distribution. *(3 Points)*

(d) Consider the differentiable function $f(x) = \sin x \in L^1_{\text{loc}}(\mathbb{R})$. Recall the definition of the distribution F_f given prior to Lemma 2.9. Show that $(F_f)' = F_{f'}$ for this example. *(2 Points)*

(e) Consider the line $L = \{y = 1\} \subset \mathbb{R}^2$. Show that

$$G(\varphi) := \int_L \varphi\, d\sigma$$

defines a distribution in $\mathcal{D}'(\mathbb{R}^2)$. Note that the $d\sigma$ indicates this is an integration over the submanifold L. Does there exist a locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(\varphi) = \int_{\mathbb{R}^2} g\, \varphi\, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R})$? (Hint. Use Lemma 2.9) *(2 Points + 2 Bonus Points)*

18. Transport and Distribution.

We have seen that every distribution is differentiable. In this question we show that every distribution is also integrable. We use this to give a solution to the inhomogeneous transport equation for distributions. Throughout this question we consider test functions $\varphi(x, t)$ in $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ and distributions in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ unless stated otherwise. First we prepare some results about test functions. Some parts of this question are difficult, so try your best and don't be discouraged.

(a) Show that

$$(\mathcal{I}\varphi)(x) := \int_{\mathbb{R}} \varphi(x, t)\, dt$$

belongs to $C_0^\infty(\mathbb{R}^n)$. *(2 Points)*

(b) Show for any distribution $H \in \mathcal{D}'(\mathbb{R}^n)$ that $F : \varphi \mapsto H(\mathcal{I}\varphi)$ is a distribution in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. *(2 Points)*

(c) Define the subset $\mathcal{Z} = \ker \mathcal{I} = \{\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \mid \mathcal{I}\varphi \equiv 0\}$ and the operator

$$(\mathcal{P}\varphi)(x,t) := \int_{-\infty}^{t} \varphi(x,s)\, ds.$$

Show that $\mathcal{P}\varphi$ is a test function if and only if $\varphi \in \mathcal{Z}$. Moreover, show that if $\varphi \in \mathcal{Z}$ then $\mathcal{P}\varphi$ is the unique test function ψ with $\partial_t \psi = \varphi$. *(3 Points)*

(d) Let χ_0 be a test function that does not depend on x with $\int_{\mathbb{R}} \chi_0(t)\, dt = 1$. For any test function φ, define

$$\tilde{\varphi}(x,t) := \varphi(x,t) - (\mathcal{I}\varphi)(x)\chi_0(t).$$

Show that $\tilde{\varphi} \in \mathcal{Z}$. *(1 Point)*

Part (d) gives a decomposition of any test function into a derivative of a test function and the product of test functions that are constant in t and x. Now we are ready to show that distributions can be 'integrated' with respect to t.

(e) Suppose that U and F are two distributions such that $\partial_t U = F$. Why must $U(\partial_t \varphi) = -F(\varphi)$ for any test function φ? *(1 Point)*

(f) Suppose we are given a distribution F. Prove that, for any $G \in \mathcal{D}'(\mathbb{R}^n)$, the following formula defines a distribution such that $\partial_t U = F$: *(1 Point + 2 Bonus Points)*

$$U(\varphi) := -F(\mathcal{P}\tilde{\varphi}) + G(\mathcal{I}\varphi).$$

Thus we see that every distribution F has many t-antiderivatives. The distribution G plays the role of the integration constant, in the same way that for ordinary functions $\partial_t(f(x,t) + g(x)) = \partial_t f$. It turns out that the converse is also true, every t-antiderivative of F has this form. This is essentially proved in Exercise 2.10(3) from the lecture script.

Finally, we show that the inhomogeneous transport equation for distributions $\partial_t U + b \cdot \nabla U = F$ is always solvable.

(g) Let $(\mathcal{T}_b\varphi)(x,t) := \varphi(x - bt, t)$ be the translation operator. For any distribution U define a distribution $\tilde{U} : \varphi \mapsto U(\mathcal{T}_b\varphi)$. Notice that although $\tilde{U}(\varphi) = U(\mathcal{T}_b\varphi)$ for any test function φ, their derivatives are subtly distinct in a way that can be hard to express in notation, namely

$$\partial_i U(\mathcal{T}_b\varphi) = -U(\partial_i(\mathcal{T}_b\varphi)), \qquad \partial_i \tilde{U}(\varphi) = -\tilde{U}(\partial_i\varphi) = -U(\mathcal{T}_b(\partial_i\varphi)).$$

Prove that $\partial_t \tilde{U}(\varphi) = \partial_t U(\mathcal{T}_b\varphi) + b \cdot \nabla U(\mathcal{T}_b\varphi)$. *(2 Points)*

(h) Finally, let F be any distribution. Use parts (f) and (g) to give a solution to the inhomogeneous transport equation. *(2 Bonus Points)*

We have not addressed the question of uniqueness; this is ultimate aim of Exercise 2.10 in the lecture script, which finds every solution to the homogeneous equation. Since the difference of two solutions to the inhomogeneous equation is a solution to the homogeneous equation, this suffices.

Neither this question or 2.10 really shows you what it means to specify the 'initial value' of a distributional PDE, though it is hinted at by the G above and 2.10(4). I leave this as a challenge to you. But if we contrast the solutions we have found here to the results of Section 1 (there is a solution for every *differentiable* function g), we see that distributions allows us to have non-differentiable solutions to PDEs in a rigorous way.

19. Preparing the Mean Value Theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$, and $\partial B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$ for $r > 0$. Show that the function

$$F(r) := \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) \, \mathrm{d}\sigma(x)$$

converges to $f(x_0)$ as $r \to 0$. *(4 Points)*