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Introduction to Partial Differential Equations
Exercise sheet 5

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12. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the first Green formula

$$\int_\Omega v \triangle u \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \nabla u \cdot N \, d\sigma.$$

(3 points)

(b) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v \triangle u - u \triangle v) \, dx = \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \, d\sigma.$$

(1 points)

(c) Suppose further that v has compact support in Ω. Prove that

$$\int_\Omega v \triangle u \, dx = \int_\Omega u \triangle v \, dx$$

(1 points)

Solution.

(a) The final term of the formula can have the divergence theorem applied to it:

$$-\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \nabla u \cdot N \, d\sigma = \int_\Omega -\nabla u \cdot \nabla v + \nabla \cdot (v \nabla u) \, dx$$

If g is a scalar-valued function and F is a vector-valued function, recall the product rule (or derive it for yourself): $\nabla \cdot (gF) = (\nabla g) \cdot F + g \nabla \cdot F$. Applying this formula here, and remembering that the Laplacian is the divergence of the gradient, gives

$$\int_\Omega -\nabla u \cdot \nabla v + \nabla \cdot (v \nabla u) \, dx = \int_\Omega -\nabla u \cdot \nabla v + \nabla v \cdot \nabla u + v \Delta u \, dx = \int_\Omega v \Delta u \, dx.$$

(b) The second Greens formula is simply a symmetrised version of the first:

$$\begin{aligned}\int_\Omega (v \triangle u - u \triangle v) \, dx &= -\int_\Omega (\nabla u \cdot \nabla v - \nabla v \cdot \nabla u) \, dx + \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \, d\sigma \\ &= \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \, d\sigma.\end{aligned}$$

(c) Since v has compact support, it and its derivatives must vanish on $\partial\Omega$. Therefore the right hand side of (b) is zero. The result follows.

13. The Black Hole.

Consider $\mathbb{R}^3$, a ball $B_r = \{x^2 + y^2 + z^2 \leq r^2\}$ and the function $g(x, y, z) = -(x^2 + y^2 + z^2)^{-0.5}$.

(a) Compute the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

where N is the outward pointing normal. Observe it that does not depend on the radius r.

(3 points)

(b) Can you apply the divergence theorem to this integral? Why or why not? *(1 point)*

(c) Compute the Laplacian of g. *(2 point)*

(d) Let $r < R$ and let $\Omega = B_R \setminus B_r$. The boundary of Ω has two components, namely ∂B_R and ∂B_r. Apply the divergence theorem to Ω with $f = \nabla g$. How does this relate to part (a)?

(3 points)

(e) Generalise the previous part to prove for any compact region $\Omega \subset \mathbb{R}^3$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \, d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

Solution.

(a) Let $\mathbf{x} \in \mathbb{R}^3$. The outward pointing unit normal of the ball of radius r is $N = \mathbf{x}/|\mathbf{x}|$, where $|\mathbf{x}| = r$ is constant on ∂B_r. To compute the gradient of $g = -|\mathbf{x}|^{-1}$, we need the partial derivatives

$$\partial_i g = \frac{1}{2} |\mathbf{x}|^{-3} \, 2x_i = |\mathbf{x}|^{-3} \, x_i,$$

so the gradient is $\nabla g = |x|^{-3}\mathbf{x}$. Together we then have

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma = \int_{\partial B_r} |\mathbf{x}|^{-4} \, \mathbf{x} \cdot \mathbf{x} \, d\sigma = \int_{\partial B_r} |\mathbf{x}|^{-2} \, d\sigma = r^{-2} \int_{\partial B_r} 1 \, d\sigma.$$

Now, the integral above is just the surface area of the sphere ∂B_r, ie $4\pi r^2$, and therefore the value is 4π.

(b) You cannot apply the divergence theorem to this integral (directly) because $f = \nabla g$ has a singularity at the origin. The divergence theorem applies to functions f which are continuous and differentiable on their domain.

(c) Beginning with the partial derivatives $\partial_i g = |\mathbf{x}|^{-3} x_i$, we differentiate again

$$\begin{aligned}
\partial_i^2 g &= \partial_i(|\mathbf{x}|^{-3} \, x_i) = -3|\mathbf{x}|^{-5} \, x_i^2 + |\mathbf{x}|^{-3} \\
\Delta g &= \partial_1^2 g + \partial_2^2 g + \partial_2^2 g \\
&= -3|\mathbf{x}|^{-5} \, x^2 + |\mathbf{x}|^{-3} - 3|\mathbf{x}|^{-5} \, y^2 + |\mathbf{x}|^{-3} - 3|\mathbf{x}|^{-5} \, z^2 + |\mathbf{x}|^{-3} \\
&= -3|\mathbf{x}|^{-5} \, (x^2 + y^2 + z^2) + 3|\mathbf{x}|^{-3} = 0.
\end{aligned}$$

This holds at all points $\mathbf{x} \neq 0$ where g is defined.

(d) When we consider ∂B_R and ∂B_r as the boundary of Ω, then the outward normals are $\mathbf{x}/|\mathbf{x}|$ and $-\mathbf{x}/|\mathbf{x}|$ respectively. So then

$$\begin{aligned}\int_\Omega \nabla\cdot(\nabla g)\, d\mu &= \int_{\partial B_R} \nabla g\cdot(\mathbf{x}/|\mathbf{x}|)\, d\sigma + \int_{\partial B_r} \nabla g\cdot(-\mathbf{x}/|\mathbf{x}|)\, d\sigma \\ &= \int_{\partial B_R} \nabla g\cdot(\mathbf{x}/|\mathbf{x}|)\, d\sigma - \int_{\partial B_r} \nabla g\cdot(\mathbf{x}/|\mathbf{x}|)\, d\sigma\end{aligned}$$

If we calculate further with the left hand side we see that it is zero on Ω, because $\nabla\cdot(\nabla g) = \Delta g = 0$. It follows then that

$$\int_{\partial B_R} \nabla g\cdot N\, d\sigma = \int_{\partial B_r} \nabla g\cdot N\, d\sigma$$

for all $R > r > 0$. This explains (a).

(e) Suppose first that $(0,0)\notin\Omega$. We have already seen that the Laplacian of g is zero, and we can immediately conclude from the divergence theorem that the integral over $\partial\Omega$ is zero too.

If however $(0,0)$ lies in the interior of Ω, we must first excise it. There exists some small ϵ such that $B_\epsilon\subset\Omega$. Then

$$\begin{aligned}\int_{\partial\Omega} \nabla g\cdot N\, d\sigma &= \int_{\partial B_\epsilon} \nabla g\cdot N\, d\sigma + \left(\int_{\partial\Omega} \nabla g\cdot N\, d\sigma - \int_{\partial B_\epsilon} \nabla g\cdot N\, d\sigma\right) \\ &= 4\pi + \int_{\Omega\setminus B_\epsilon} \Delta g\, dx \\ &= 4\pi.\end{aligned}$$

14. Convoluted.

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\, dy.$$

(a) Let $f_n(x) = n$ for $0 \le x \le n^{-1}$ and 0 otherwise. Show that the following bounds hold

$$\inf_{y\in I_n} g(y) \le (g * f_n)(0) \le \sup_{y\in I_n} g(y).$$

(3 Points)

(b) Suppose now that g is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n\to\infty$. *(3 Points)*

(c) (Optional) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation.

Solution.

(a) Let $I_n = [-n^{-1}, 0]$ for brevity. We compute

$$\begin{aligned}(g * f_n)(0) &= \int_{-\infty}^{\infty} g(y) f_n(-y) = \int_{-n^{-1}}^{0} g(y) \times n \\ &\leq n \int_{-n^{-1}}^{0} \sup_{y \in I_n} g(y) = n \sup_{y \in I_n} g(y) \times n^{-1} \\ &= \sup_{y \in I_n} g(y)\end{aligned}$$

In a similar manner, we see that $(g * f_n)(0)$ is bound below by $\inf_{y \in I_n} g(y)$.

(b) Clearly $\sup_{y \in I_n} g(y) \geq g(0)$. On the other hand, choose any $\epsilon > 0$. By the continuity of g, there exists $\delta > 0$ such that $|g(y) - g(0)| < \epsilon$ for all $y \in (-\delta, \delta)$. Choose N such that $N^{-1} < \delta$. That means for all $y \in I_N$ we have $|g(y) - g(0)| < \epsilon$. For all $n > N$ the interval $[-n^{-1}, 0]$ is a subset of $[-N^{-1}, 0]$. It follows that

$$\sup_{y \in I_n} g(y) \leq \sup_{y \in I_N} g(y) < \sup_{y \in I_N} (g(0) + \epsilon) = g(0) + \epsilon.$$

These two inequalities together say that $\forall \epsilon > 0 \, \exists N \, \forall n > N$ it holds that

$$|\sup_{y \in I_n} g(y) - g(0)| < \epsilon.$$

This is the definition of $\sup_{y \in I_n} g(y) \to g(0)$. The same argument shows that $\inf_{y \in I_n} g(y) \to g(0)$ also. By the sandwich rule/squeeze rule, the result follows.

(c) Formal bilinearity follows from the linearity of the integral and the bilinearity of the product of functions. The smoothness and compact support of all functions involved means that the integrals always exist.

Commutativity:

$$f * g(x) = \int_{-\infty}^{\infty} f(y) g(x - y) \, dy = \int_{\infty}^{-\infty} f(x - z) g(z) \, (-dz) = g * f(x).$$

Associativity

$$\begin{aligned}f * (g * h)(x) &= \int_{-\infty}^{\infty} f(y)(g * h)(x - y) \, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z) h((x - y) - z) \, dz \, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z) h(x - (y + z)) \, dz \, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(w - y) h(x - w) \, dy \, dw \\ &= \int_{-\infty}^{\infty} (f * g)(w) h(x - w) \, dw \\ &= (f * g) * h(x)\end{aligned}$$

15. 🚀 🚀 🚀 🚀 🚀 🚀 🚀 🚀 🚀

In economics, the Black–Scholes equation is a PDE that describes the price V of a (European-style) option which under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 point(s))

Solution. The highest derivative in the equation is second order, so this is a second order PDE. To determine which type it is, bring all the terms to one side and order them as per the general form of a second order linear PDE

$$\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 0\frac{\partial^2 V}{\partial S \partial t} + 0\frac{\partial^2 V}{\partial t \partial S} + 0\frac{\partial^2 V}{\partial t^2}\right) + \left(rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t}\right) - rV = 0.$$

From this we can read off all the coefficient functions:

$$a = \begin{pmatrix} \frac{1}{2}\sigma^2 S^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} rS \\ 1 \end{pmatrix}, \quad c = -r.$$

We see that the matrix a is positive semi-definite and in fact its kernel is one dimensional. That makes it a parabolic PDE.

16. Go with the flow.

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity ρv, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold S is the integral

$$\int_S \rho v \cdot N \, d\sigma,$$

where N is the normal of S.

(a) Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

(b) How does this relate to the form of the conservation law derived in the lectures?

(c) For liquids a common property is *incompressibility*. For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that ρ is constant. However, slightly more general model says that ρ is not globally constant, but if we follow a point $x(t)$ along the velocity field v then $\rho(x(t), t)$ is constant.
Use this description of incompressible flow to show that $\nabla \cdot v = 0$.

Solution.

(a) By defining flux in the way we have, the divergence theorem applies. Let S be a surface enclosing a volume V:

$$\int_S \rho v \cdot N \, d\sigma = \int_V \nabla \cdot (\rho v) \, dx.$$

On the other hand, the amount of substance in V is the integral of ρ. Conservation means that the (positive) change of substance should be equal to the negative of the outward flux:

$$\frac{\partial}{\partial t} \int_V \rho \, dx = - \int_S \rho v \cdot N \, d\sigma \Rightarrow \int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) \, dx = 0.$$

Since this should hold for every volume V, we conclude that the integrand must be identically zero. We will give a rigorous proof of this final step in a few weeks time.

(b) The conservation law in lectures was only for one-dimensional situations, so $\nabla \cdot$ is the same as ∂_x. In the previous part we could also have an arbitrary flux function f instead of only the form ρv and still apply the same working to arrive at

$$\frac{\partial \rho}{\partial t} + \nabla \cdot f = 0 \Rightarrow \dot{\rho} + \partial_x f = 0.$$

If the flux function only depends on the density ρ, and not on the coordinate, then we arrive exactly at the form in Theorem 1.10.

(c) The condition of being constant along a flow is very similar to the idea behind the method of characteristics. If we take the total derivative

$$0 = \frac{d}{dt} \rho = \nabla \rho \cdot \dot{x} + \dot{\rho} = \nabla \rho \cdot v + \dot{\rho}.$$

On the other hand, we can expand the conservation law

$$0 = \dot{\rho} + \nabla \cdot (\rho v) = \dot{\rho} + \nabla \rho \cdot v + \rho \nabla \cdot v = \rho \nabla \cdot v.$$

So either there is no substance (which is trivial) or $\nabla \cdot v = 0$ as required.