Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 5

28th October, 2021

12. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the first Green formula

$$\int_\Omega v\triangle u \; dx = -\int_\Omega \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} v\nabla u \cdot N \; d\sigma.$$

(3 points)

(b) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v\triangle u - u\triangle v) \; dx = \int_{\partial\Omega} (v\nabla u - u\nabla v) \cdot N \; d\sigma.$$

(1 points)

(c) Suppose further that v has compact support in Ω. Prove that

$$\int_\Omega v\triangle u \; dx = \int_\Omega u\triangle v \; dx$$

(1 points)

13. The Black Hole.

Consider $\mathbb{R}^3$, a ball $B_r = \{x^2 + y^2 + z^2 \leq r^2\}$ and the function $g(x, y, z) = -(x^2 + y^2 + z^2)^{-0.5}$.

(a) Compute the integral

$$\int_{\partial B_r} \nabla g \cdot N \; d\sigma$$

where N is the outward pointing normal. Observe it that does not depend on the radius r.

(3 points)

(b) Can you apply the divergence theorem to this integral? Why or why not? *(1 point)*

(c) Compute the Laplacian of g. *(2 point)*

(d) Let $r < R$ and let $\Omega = B_R \setminus B_r$. The boundary of Ω has two components, namely ∂B_R and ∂B_r. Apply the divergence theorem to Ω with $f = \nabla g$. How does this relate to part (a)?

(3 points)

(e) Generalise the previous part to prove for any compact region $\Omega \subset \mathbb{R}^3$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \; d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

14. Convoluted.

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

(a) Let $f_n(x) = n$ for $0 \le x \le n^{-1}$ and 0 otherwise. Show that the following bounds hold

$$\inf_{y \in I_n} g(y) \le (g * f_n)(0) \le \sup_{y \in I_n} g(y).$$

(3 Points)

(b) Suppose now that g is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n \to \infty$. *(3 Points)*

(c) (Optional) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation.

15. 🚀 🚀 🚀 🚀 🚀 🚀 🚀 🚀 🚀

In economics, the Black–Scholes equation is a PDE that describes the price V of a (European-style) option which under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 point(s))

16. Go with the flow.

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity ρv, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold S is the integral

$$\int_S \rho v \cdot N\,d\sigma,$$

where N is the normal of S.

(a) Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

(b) How does this relate to the form of the conservation law derived in the lectures?

(c) For liquids a common property is *incompressibility*. For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that ρ is constant. However, slightly more general model says that ρ is not globally constant, but if we follow a point $x(t)$ along the velocity field v then $\rho(x(t), t)$ is constant.

Use this description of incompressible flow to show that $\nabla \cdot v = 0$.