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Introduction to Partial Differential Equations
Exercise sheet 4

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9. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 point(s))

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.
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(4 point(s))

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(2 point(s))*

Solution.

(a) By computation

$$\frac{d}{ds}z(s) = \nabla u(x(s)) \cdot \dot{x}(s) = \nabla u(x(s)) \cdot b(x(s)) = -c(x(s))u(x(s)) = -c(x(s))z(s).$$

(b) We follow the working at the beginning of Section 1.5 of the lecture script, specialising the argument to this particular case. As there, we have

$$\frac{dp}{ds} = \text{Hess}(u)\dot{x} = \left(\sum_j \partial_i \partial_j u \, \dot{x}_j \right),$$

where $\text{Hess}(u)$ is the matrix of second derivatives of u. The total derivative of F with respect to x is

$$\begin{aligned} 0 &= \partial_p F \cdot \partial_i p + \partial_z F \partial_i z + \partial_i F \\ &= b \cdot \partial_i p + \partial_z F \, p_i + \partial_i F \\ 0 &= \text{Hess}(u) b + \partial_z F \, p + \nabla F. \end{aligned}$$

If we suppose that the characteristic has the property that $\dot{x} = b(z,x)$, then

$$\dot{p} = -\partial_z F\, p - \nabla F.$$

So $p(s)$ is described by an ODE and the assumption about $\dot{x}$ does not involve p. Finally,

$$\dot{z} = \nabla u \cdot \dot{x} = p \cdot \dot{x} = p \cdot b = -c(z,x)$$

since by assumption $F(p(s), z(s), x(s)) = 0$. This also does not depend on p. Hence we have the ODE for the characteristics, and the $\dot{x}$ and $\dot{z}$ equations do not depend on p.

(c) The inhomogeneous transport equation is defined by

$$F(p,z,x) = \tilde{b} \cdot p - f(\tilde{x})$$

where $\tilde{x} = (x,t)$ and $\tilde{b} = (b,1)$ in $\mathbb{R}^{n+1}$. From the equations we have just derived, we see that $\dot{\tilde{x}}(s) = (b,1)$ tells us that the characteristic lines are straight lines $\tilde{x}(s) = (bs + x_0, s)$. Or in non-parametric form $x = bt + x_0$. The next ODE is $\dot{z}(s) = f(x(s), s) = f(x_0 + bs, s)$. This too can be directly integrated now

$$z(t) - z(0) = \int_0^t \dot{z}(s)\, ds = \int_0^t f(x_0 + bs, s)\, ds = \int_0^t f(x - bt + bs, s)\, ds.$$

Together with the initial condition $z(0) = u(x(0), 0) = u(x_0, 0) = g(x - bt)$ this is exactly the solution that we found previously.

10. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $\partial_1 u + \partial_2 u = u^2$ on the plane with boundary condition $u(x_1, 0) = g(x_1)$.

(4 point(s))

(b) $x_1 \partial_2 u - x_2 \partial_1 u = u$ on the domain $x_2 > 0$, with boundary condition $u(0, x_2) = g(x_2)$.

(4 point(s))

(c) $u \partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2} x_1$.

(5 point(s))

Solution. These PDEs are all of the linear type of the previous question, so we can use the ODEs for the characteristics that we have already derived.

(a) Putting this PDE in the above form gives $(1,1) \cdot p - z^2 = 0$, making $b(z,x) = (1,1)$ a constant function and $c(z,x) = z^2$ a function of z alone. Thus the characteristics are just the lines $x(s) = (x_{10} + s, x_{20} + s) = (a + s, s)$. Given a point (x_1, x_2) it lies on the characteristic at $s = x_2$ with $a = x_1 - x_2$.

The value of the function u is described along the characteristics by z, which has the ODE $z' = -c = z^2$. This has solution $z(s) = -(s+\sigma)^{-1}$ for integration constant $\sigma = -z(0)^{-1}$. Therefore

$$u(x,t) = \frac{-1}{s - u(a,0)^{-1}} = \frac{-1}{x_2 - u(x_1 - x_2, 0)^{-1}} = \frac{-1}{x_2 - g(x_1 - x_2)^{-1}}$$

(b) This PDE is $(-x_2, x_1) \cdot p - z = 0$. The system of ODEs therefore reads in part

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1,$$

which is the well know system solved by the sinusoidal functions. From the boundary $(0, x_2)$ we see that $x_1 = 0$ when $s = 0$. Therefore $x_2 = r\sin(s + \pi/2)$ and $x_1 = r\cos(s + \pi/2)$ for a constant r, which identifies the characteristic. The ODE describing the values of u is $\dot{z} = z$, so $u(x(s)) = e^s u(x(0)) = e^s u(0, r) = e^s g(r)$. We must now solve for the parameters r, s in terms of the point x:

$$r = |x|, s = \arg(x_1, x_2) - \frac{\pi}{2}$$

where arg gives the angle of the point in the upper half-plane in the range $[0, \pi]$. Together this gives

$$u(x) = e^{\arg(x_1,x_2) - \frac{\pi}{2}} g(|x|).$$

(c) This PDE, $F = (z, 1) \cdot p - 1$ is a little different to the others, because of the z in the coefficients of p. This creates a linkage in the system of ODEs:

$$\dot{x}_1 = z, \quad \dot{x}_2 = 1, \quad \dot{z} = 1.$$

Fortunately, we can solve for z first this time quite easily: $z(s) = s + z(0)$. Then $x(s) = (\frac{1}{2}s^2 + sz(0) + a, s + a)$, using the fact that the initial boundary is (a, a). Hence we can say that $z(0) = u(x(0)) = u(a, a) = \frac{1}{2}a$ and $a = x_2 - s$, which allows us to solve for s:

$$\begin{aligned} x_1 &= \frac{1}{2}s^2 + s\frac{1}{2}(x_2 - s) + x_2 - s \\ x_1 - x_2 &= \frac{1}{2}x_2 s - s \\ s &= \frac{2x_1 - 2x_2}{x_2 - 2}. \end{aligned}$$

Finally, what we are interested in is the value of the solution u on these curves, and $u(x(s)) = z(s) = s + z(0) = s + \frac{1}{2}(x_2 - s)$, ie

$$u(x) = \frac{1}{2}x_2 + \frac{1}{2}\frac{2x_1 - 2x_2}{x_2 - 2}.$$

11. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C xy \, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 bonus point(s))*

(b) Take $A = K = C$ in Definition 2.3. Consider the parametrisation $\Phi : (0, 2\pi) \to C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral using this parametrisation. *(2 point(s))*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1 - x^2})$. Compute the integral using these parametrisations. *(2 point(s))*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. [Hint. The easiest way is to use two parametrisations similar to part (b)]

(e) Compute this integral using the divergence theorem. *(3 point(s))*

Solution.

(a) This depends somewhat on the definition of integral that you are using. In Lebesgue integration, sets of measure zero can not contribute to the final value, and a finite collection of points is measure zero in dimensions 1 and higher. With Darboux or Riemann integrations, these are defined initially on closed sets only. They are extended to open sets, or in this case punctured neighbourhoods by taking a limit of closed sets. Continuity of the integrand is sufficient then to ensure there is no difference.

(b) Using the previous part, we know that we can integrate with the parametrisation over $U = (0, 2\pi)$, and simply ignore the point $(1, 0) \in C$ that is not covered because that does not affect the value of the integral.

We must also calculate the area-element factor. The coordinate maps $\Phi : (0, 2\pi) \subset \mathbb{R} \to \mathbb{R}^2$, so its derivative is size 2×1, namely $(-\sin t, \cos t)^T$. The factor therefore is

$$\det \begin{bmatrix} -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \det \begin{bmatrix} 1 \end{bmatrix} = 1$$

We can now compute the integral

$$\int_0^{2\pi} \cos t \times \sin t \times 1 \, dt = \int_0^{2\pi} \frac{1}{2} \sin 2t \, dt = -\frac{1}{4} \cos 2t \Big|_0^{2\pi} = 0.$$

(c) Here we have that $\Phi_1' = (1, -x(1 - x^2)^{-0.5})^T$. So

$$\int_{U_1} xy \, d\sigma = \int_{-1}^{1} x \times \sqrt{1 - x^2} \times \sqrt{1 + x^2(1 - x^2)^{-1}} \, dx$$
$$= \int_{-1}^{1} x \, dx = \frac{1}{2} x^2 \Big|_{-1}^{1} = 0.$$

And likewise for U_2.

(d) As hinted at, start with the parametrisation Φ defined in part (b). Now take a bump function h on $(0, 2\pi)$ such that it is identically 1 on $[\pi/2, 3\pi/2]$ and has compact support K strictly contained in the interval. Now, we need a second coordinate chart to cover the point $t = 0$, ie $(1, 0) \in C$. For this we use $\Psi : (-\pi/2, \pi/2)$ given by $\Psi(t) = (\cos t, \sin t)$. Because it has the same formula, the area-element of Ψ is also 1.

Notice that $V_1 = C \setminus \{(1, 0)\}$, $h_1 = h$, $V_2 = C \cap \{x > 0\}$, and $h_2 = 1 - h$ is a partition of unity for C, and that Φ and Ψ are coordinates for the two set respectively. Hence

$$\begin{aligned}
\int_C x \, d\sigma &= \int_0^{2\pi} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t)) \cos t \, dt \\
&= \int_{\pi/2}^{3\pi/2} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t)) \cos t \, dt \\
&= \int_{\pi/2}^{3\pi/2} \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t) + h(t)) \cos t \, dt \\
&= \int_{-\pi/2}^{3\pi/2} \cos t \, dt,
\end{aligned}$$

which comes out to the same calculation as in part (b). Thus we see that a partition of unity is a technical tool to divide an integral into pieces without worrying about overlaps and/or missing points, but not so practical for calculation.

(e) To apply the divergence theorem, we recognise C as the boundary of the disc $\Omega = \{x^2+y^2 \leq 1\}$, with the outward pointing normal $N = (x, y)$. (Because this it is the unit circle, this normal N is already unit length.) We now need to write the integrand x in the form $f \cdot N = (f_1, f_2) \cdot N = xf_1 + yf_2$. We see that $f = (y, 0)$ fits. The divergence of f is

$$\nabla \cdot f = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} 0 = 0.$$

Therefore

$$\int_C xy \, d\sigma = \int_{\partial\Omega} f \cdot N \, d\mu = \int_\Omega \nabla \cdot f \, d\mu = \int_\Omega 0 \, d\mu = 0.$$