Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 4

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9. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 point(s))

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.
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(4 point(s))

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(2 point(s))*

10. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $\partial_1 u + \partial_2 u = u^2$ on the plane with boundary condition $u(x_1, 0) = g(x_1)$.

(4 point(s))

(b) $x_1 \partial_2 u - x_2 \partial_1 u = u$ on the domain $x_2 > 0$, with boundary condition $u(0, x_2) = g(x_2)$.

(4 point(s))

(c) $u\partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2}x_1$.

(5 point(s))

11. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C xy \, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 bonus point(s))*

(b) Take $A = K = C$ in Definition 2.3. Consider the parametrisation of the circle $\Phi : (0, 2\pi)$, $t \mapsto (\cos t, \sin t)$. Compute the integral in this parametrisation. *(2 point(s))*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1 - x^2})$. Compute the integral in these parametrisations. *(2 point(s))*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. [Hint. The easiest way is to use two parametrisations similar to part (b)]

(e) Compute this integral using the divergence theorem. *(3 point(s))*