Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 3

21st September, 2021

6. Racecar go broooom.

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) What properties do you think that f should have? Does $f(u) = s_m u \cdot (1 - u/u_m)$ have these properties? *(2 point(s))*

(b) Find a function f that meets your conditions, or use the f from the previous part, and write down a PDE to describe the traffic flow. *(1 point(s))*

(c) Find all solutions that are constant in time. *(2 point(s))*

(d) Consider the situation of the start of a race: to the left of the starting line, the racecars are queued up at half of the maximum density (ie $0.5u_m$). To the right of the starting line, the road is empty. Now, at time $t = 0$, the race begins. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(5 point(s))*

7. All you can eat. Consider the the scalar conservation PDE for $f(u) = \frac{1}{3}u^3$ with the initial condition $u_0(x) = x$, similar to Exercise 4.

(a) Determine the characteristics of this problem. *(1 point(s))*

(b) Up until which time does there exist a strong solution? *(2 point(s))*

(c) Consider now the same PDE and initial condition, but with the domain $(t, x) \in [0, \infty) \times [0, \infty)$. Find the solution. Is it unique? *(4 point(s))*

(d) Consider now the same PDE and initial condition, but with the domain $(t, x) \in [0, \infty) \times [1, \infty)$. Find two solutions. *(3 point(s))*

8. Method of characteristics for an Inhomogeneous PDE Use the method of characteristics to solve the following *inhomogeneous* PDE. Note, the function u will *not* be constant along the characteristic, but its value along the characteristic will be determined by its initial value.

$$x\partial_x u + y\partial_y u = 2u$$

on the domain $x \in \mathbb{R}, y > 0$, with initial condition $u(x, 1) = x$.

(5 point(s))

Solutions are due on Monday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de` as a pdf. One possibility is to write your solutions neatly by hand and then scan them with your phone. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.