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Introduction to Partial Differential Equations
Exercise sheet 2

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3. Inhomogeneous Transport Equation. First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous equation

$$\frac{du}{dt} = f(t).$$

(a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

(b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique, if you haven't seen the lecture/read the script.) *(2 points)*

Solution.

(a) A solution is given by the integral $\int_0^t f(s)\, ds$.

(b) A solution to the initial value problem is

$$u(t) = c + \int_0^t f(s)\, ds.$$

This is a sum of a solution to the homogeneous equation that satisfies the initial value and a solution to the inhomogeneous equation that has an initial value of zero.

Suppose that there is another solution v. Then by linearity $u - v$ solves the homogeneous equation $\frac{du}{dt} = 0$ and therefore is a constant. But we know these solutions have the same initial value and hence $u - v = 0$ for all time.

(c) We compute the t and x derivatives of the integral term

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^t f(x + (s-t)b, s)\, ds &= f(x + (t-t)b, t)\, ds + \int_0^t (-b) \cdot \nabla f(x + (s-t)b, s)\, ds \\ &= f(x,t)\, ds - b \cdot \int_0^t \nabla f(x + (s-t)b, s)\, ds\end{aligned}$$

$$\nabla \int_0^t f(x + (s-t)b, s)\, ds = \int_0^t \nabla f(x + (s-t)b, s)\, ds.$$

The sum is equal to $f(x,t)$ as required. We find the initial value of this function by substituting $t = 0$. But then we have $\int_0^0$ which is always 0.

(d) Because this is a linear equation, the difference between two solutions of the inhomogeneous equation is a solution of the homogeneous equation with zero initial value. We have seen in lectures that the solution to the homogeneous transport equation with initial value is unique. Therefore any two solutions must be equal.

4. Royale with Cheese

Recall Burgers' equation from Example 1.6 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $u_0(x) = x$.

(a) According to Theorem 1.5, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point(s))*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 point(s))*

(c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution. *(2 point(s))*

(d) Finally, derive the following solution to the initial value problem:

$$u(x, t) = \frac{x}{1 + t}.$$

(2 point(s))

(e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 point(s))*

(f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point(s))*

Solution.

(a) The condition in the theorem depends on the bound $f''(u_0(x))u_0'(x) \geq -\alpha$. For this equation, both $f''(u) = 1$ and $u_0'(x) = 1$, so the product is bound below easily by $\alpha = 0$. Hence the theorem says that the unique solution exists for all time.

(b) By the chain rule,

$$\frac{d}{ds}u(x(s), t(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{dt}{ds}\dot{u} + \frac{dx}{ds}\partial_x u.$$

If we compare this to the PDE, then we see that we should choose $t' = 1$, i.e. $t = s$, and $x' = u$.

(c) The characteristics are the rays $x = x_0 t + x_0$ for $x_0 \in \mathbb{R}$. The solution takes the value x_0 on the corresponding ray. We can see that the 'mass' (the conserved quantity) is flowing away from the origin.

(d) For the chosen characteristics we know that $\frac{d}{dt}u(x(t), t) = 0$ by the choice of the constraints on the path. Hence u is constant on this path and equal to $u_0(x(0))$. The ODE $x'(s) = u(x(s), s) = u_0(x(0))$ is now easily to integrate, giving

$$x(t) = x_0 + u_0(x_0)t = x_0 + x_0 t = x_0(1 + t).$$

In other words, we know that for all x_0 and t

$$u(x(t), t) = u(x_0(1 + t), t) = u_0(x_0) = x_0.$$

We have now found the solution, but to make it clearer, let $\tilde{x} = x_0(1 + t)$. Then

$$u(\tilde{x}, t) = \frac{\tilde{x}}{1 + t}.$$

(e) The solution is clearly well defined for $t > 0$ and all x. We compute

$$\dot{u} = -\frac{x}{(1 + t)^2}, \qquad \partial_x u = \frac{1}{1 + t}.$$

Thus we see that the partial differential equation is solved. As too is the initial value $u(x, 0) = x = u_0(x)$.

(f) Because such PDEs resemble the chain rule. Hence we can identity the derivatives $x_i'(s)$ of the path with the coefficient functions in PDE and reduce it to a system of ODEs.

5. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.7. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x, t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x, t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x, t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x,t)-plane for this function. *(2 point(s))*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(3 point(s))*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(3 point(s))*

Solution.

(a) Refer to lecture script.

(b) The characteristics for $t < 1$ are described in the lecture script

$$x = \begin{cases} t + x_0 & \text{for } x_0 \leq 0 \\ t(1 - x_0) + x_0 & \text{for } 0 < x_0 \leq 1 \\ x_0 & \text{for } 1 < x_0. \end{cases}$$

However, for $t > 1$ there are two regions: $x > 1$ and $x < 1$. In the former the characteristics continue to be horizontal lines. In the lower region they are lines with gradient 1.

(c) The discontinuity is for $t > 1$ when the solution jumps from 0 to 1. This occurs on the line $y(t) = 1$. Hence $\dot{y} = 0$. On the other hand, $f(u) = \frac{1}{2}u^2$, the value of u on the upper side of the discontinuity is $v^r(1^+, t) = 0$ and $v^l(1^-, t) = 1$. The right hand side of the Rankine-Hugonoit condition is then

$$\frac{\frac{1}{2} \cdot 0^2 - \frac{1}{2} \cdot 1^2}{0 - 1} = \frac{1}{2}.$$

This shows that v does not fulfil the condition.

(d) We know that mass is conserved away from the discontinuity. Therefore we jus need to know how much is being lost across the discontinuity. This is easy to compute in this case because the discontinuity is not moving, $y(t) = 1$. So first we compute the amount of mass in some interval containing the discontinuity, say $x \in [0, 2]$, for $t > 1$:

$$\int_0^2 v(x,t)\, dx = \int_0^1 1\, dx + \int_1^2 0\, dx = 1.$$

So there is a constant amount of mass in the interval. And then we compute how much mass is moving in and out of this interval:

$$f(u(0,t)) - f(u(2,t)) = f(1) - f(0) = \frac{1}{2}.$$

So there is a constant inflow of 0.5 units of mass per unit of time. Hence the system must be losing 0.5, because this inflow is not increasing the amount in the interval.

Solutions are due on Monday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de` as a pdf. One possibility is to write your solutions neatly by hand and then scan them with your phone. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.