Martin Schmidt
Ross Ogilvie

# Introduction to Partial Differential Equations
## Exercise sheet 2

14th September, 2021

**3. Inhomogeneous Transport Equation.** First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous equation

$$\frac{du}{dt} = f(t).$$

**(a)** Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

**(b)** For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\, ds.$$

**(c)** Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

**(d)** Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique, if you haven't seen the lecture/read the script.) *(2 points)*

**4. Royale with Cheese**

Recall Burgers' equation from Example 1.6 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $u_0(x) = x$.

**(a)** According to Theorem 1.5, there is a unique $C^1$ solution to this initial value problem, at least when $t$ is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point(s))*

**(b)** Suppose that $u$ is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of $u$. What is the $s$ derivative of $u$ along this path? What constraints should we place on the derivatives of $x$ and $t$? *(2 point(s))*

**(c)** On an $(x,t)$-plane, draw the characteristics and describe the behaviour of this solution. *(2 point(s))*

**(d)** Finally, derive the following solution to the initial value problem:

$$u(x,t) = \frac{x}{1+t}.$$

*(2 point(s))*

**(e)** Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 point(s))*

**(f)** Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point(s))*

**5. It's just a jump to the left**

In this question we explore some other solutions to the initial value problem from Example 1.7. As we saw, for small $t$ the method of characteristics gives a unique solution

$$u_{t<1}(x,t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

**(a)** (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value $u$ should have at a point $(x, t)$. However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x,t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

**(b)** Draw the corresponding characteristics diagram in the $(x, t)$-plane for this function. *(2 point(s))*

**(c)** Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for $v$. *(3 point(s))*

**(d)** How much mass (i.e. the integral of $v$ over $x$) is being lost in the system described by $v$ for $t > 1$? *(3 point(s))*

Solutions are due on Monday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de` as a pdf. One possibility is to write your solutions neatly by hand and then scan them with your phone. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.