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Introduction to Partial Differential Equations
Exercise sheet 1

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1. **Multiindices and the Generalised Leibniz rule.** In this question we introduce multiindex notation. A *multiindex* of n variables is a vector $\gamma \in \mathbb{N}_0^n$.

   (a) Let $x = (x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$. Write out the full expression for the derivative $\partial^{(0,2,1)}$.

   (b) Why do we need to assume that partial derivatives commute for multiindex notation to be useful?

   (c) Which multiindices satisfy $|\gamma| \leq 2$ and which satisfy $\gamma \leq (0, 2, 1)$?

   (d) The generalised binomial coefficient for multiindices is defined to be

   $$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n}.$$

   One justification for calling these binomial coefficients is the following property. Let $e_j = (0, \ldots, 1, \ldots, 0)$ be the multiindex with 1 is the j-th position and 0 in all other positions. Then for any j

   $$\binom{\gamma}{\delta} = \binom{\gamma - e_j}{\delta - e_j} + \binom{\gamma - e_j}{\delta}.$$

   Prove this property.

   (e) Let $u, v : \Omega \to \mathbb{R}$ be smooth enough functions on an open subset $\Omega \subset \mathbb{R}^n$. Show for all multiindices $\gamma \in \mathbb{N}_0^n$ the following product rule:

   $$\partial^\gamma(uv) = \sum_{0 \leq \delta \leq \gamma} \binom{\gamma}{\delta} \partial^\delta u \, \partial^{\gamma-\delta} v$$

2. **Chain rule in multiple variables.** Recall the chain rule for functions of multivariable variables (Satz 10.4(iii) in Schmidt's Analysis II script): Let $f : U \subset X \to Y$ be differentiable at $x_0 \in U$ and $g : V \subset Y \to Z$ be differentiable at $f(x_0) \in f[U] \subset V$. Then $g \circ f$ is differentiable at x_0 and

   $$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

   (a) Why does this chain rule above use function composition, when the chain rule for functions of a single variable uses multiplication? i.e.

   $$\frac{d}{dx}(x^2 + 1)^3 = 3(x^2 + 1)^2 \cdot 2x. = 6x(x^2 + 1)^2.$$

   (b) Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}^n$. Express the chain rule with partial derivatives to show that

   $$\frac{d}{dt}u(x(t)) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{dx_i}{dt}.$$

   (c) Write the above formula in terms of gradients and dot products.

   (d) Consider the function $u(x, y) = x^2 + 2y$ and the polar coordinates $x = r\cos\theta, y = r\sin\theta$. Compute the radial and angular derivatives of u.

(e) Consider a scalar function $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of $2n + 1$ variables and a function $u : \mathbb{R}^n \to \mathbb{R}$. Write an expression for the derivative of $F(\nabla u(x), u(x), x)$ with respect to x_1.