all partial derivative

30a) Schwartz function decays faster than polynomials.

$f(x) = e^{-x^2} \to 0$ for all n, for $|x| \to \infty$

$f^{(k)}(x) = p_k(x) e^{-x^2}$ $\qquad x^n p_k(x) e^{-x^2} \to 0$ for $|x| \to \infty$.

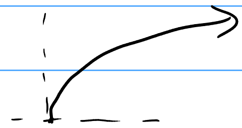
All functions of compact support. $\qquad$ Fourier transforms defined L^2

b) $f(x) = e^{-x^2}(e^{-x^2} + \sin^2 x)$

This is Schwartz basically because e^{-x^2} term.

$g = \sqrt{f} = e^{-\frac{1}{2}x^2}\sqrt{e^{-x^2} + \sin^2 x}$

Behavior as $x \to \infty$



observe $\sqrt{x}$ has ∞ derivative at $x = 0$.

$h(x) = e^{-\frac{1}{2}x^2}\sqrt{\underbrace{e^{-x^2}}_{\text{small}} + \sin^2 x}$ vanishes at $x = k\pi$

$h(k\pi)$ will have large derivative as $k \to \infty$.

c) For fixed t $\qquad \Phi(x, t) = A e^{-Bx^2} = Ae^{-B|x|^2}, x \in \mathbb{R}^n$

Martin writes $x^2 = |x|^2$, xt for dot product

Basic property of Fourier series on $S^1 = [0,1)$ $\quad e^{ix}$, $\sin x$, $e^{2\pi i x}$

k, ℓ integers.

$$\widehat{\left(e^{2\pi i \ell x}\right)}(k) = \int_{S^1} e^{-2\pi i k y} e^{2\pi i \ell y} \, dy$$

$$= \int_0^1 e^{2\pi i y(k-\ell)} \, dy$$

$$= \begin{cases} \text{if } k \neq \ell & \frac{1}{2\pi i(k-\ell)} e^{2\pi i y(k-\ell)} \Big|_0^1 = 0 \\ \text{if } k = \ell & y\Big|_0^1 = 1 \end{cases}$$

$$f(x) = \sum_{\ell \in \mathbb{Z}} a_n e^{2\pi i \ell x} = \sum \langle f, e^{2\pi i \ell y} \rangle e^{2\pi i \ell x}$$

$$= \sum \underline{\underline{\langle f, e_\ell \rangle}} e_\ell$$

Fourier Series

$$\int f(y) e^{-2\pi i k y} \, dy = a_k$$

$$\langle f, e^{2\pi i k y} \rangle \quad L^2\text{-norm.}$$

Fourier transform $\Rightarrow$ same idea on $\mathbb{R}$.

$$f = \int \underline{\underline{\tilde{f}}} e^{2\pi i k x} \, dk$$

a) In 1-D. $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ $\quad \Phi = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$

$$\hat{\Phi}(k,t) = \int_{\mathbb{R}} e^{-2\pi i k y} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} \, dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\Big(\underbrace{-2\pi i k y - \frac{y^2}{4t}}\Big) \, dy$$

$a^2y^2 \qquad 2aby \qquad b^2$

$$\frac{y^2}{4t} + 2\pi i k y + \left(\pi i k \sqrt{4t}\right)^2 - \left(\pi i k \sqrt{4t}\right)^2$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp - \left[\left(\frac{y}{\sqrt{4t}} + \pi i k \sqrt{4t} \right)^2 + \pi^2 k^2 4t \right] dy$$

$$= \frac{\exp(-4\pi^2 k^2 t)}{\sqrt{\cancel{4}\pi \cancel{t}}} \int_{\mathbb{R}} \exp(-z^2) \; \cancel{\sqrt{4}}t \, dz$$

$$z = \frac{y}{\sqrt{4t}} + \pi i k \sqrt{4t}$$

$$= \frac{\exp(-4\pi^2 k^2 t)}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

e) $\sum_{n\in\mathbb{Z}} |f(x+n)| = |f(x)| + |f(x-1)| + \sum_{n\neq 0,-1} |f(x+n)|$

x is fixed

We know f decays faster than any polynomial, eg $(x+n)^2$ in n.

$\forall \varepsilon > 0$ $\quad |(x+n)^2 f(x+n)| < \varepsilon$ as $n \to \pm\infty$

$\exists N$ so that $\forall |n| \geq N$

$$|f(x+n)| < \frac{\varepsilon}{(x+n)^2}$$

choose N so that $|x| < N$ for all $x \in \Omega$

$$\leq \underbrace{\sum_{|n|<N} |f(x+n)|}_{\text{finite.}} + \underbrace{\sum_{|n|\geq N} \frac{\varepsilon}{(x+n)^2}}_{\text{convergent}} \qquad \frac{1}{n^2}$$

f) If the boundary is periodic show the solution is periodic.

$$(\Delta - \partial_t) u = 0$$

$$u|_{t=0} = h$$

$$\partial_x (u(x+a,t)) = (\partial_x u)(x+1,t) \times 1$$

$V(x,t) = u(x+1,t) - u(x,t)$ is a sol^n to heat equation.

$v(x,0) = h(x+1) - h(x) = 0 \Rightarrow v(x,t) = 0$ by uniqueness

g) $S^1 = [0,1]$ with $f(x) = f(x+1)$

Suppose we need a sol^n to $(\Delta - \partial_t) u = 0$, $u|_{t=0} = h$

is a sol^n on all of $\mathbb{R}^n$

Then by representation $u(x,t) = \int_{\mathbb{R}} h(z) \Phi(x-z,t)\, dz$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} h(z) \Phi(x-z,t)\, dz$$

$z = y+n \qquad dz = dy \qquad z=n \quad y=0 \qquad z=n+1 \quad y=1$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 h(y) \Phi(x-(y+n),t)\, dy$$

$$= \int_0^1 h(y) \underbrace{\left[\sum_{n\in\mathbb{Z}} \Phi(x-(y+n),t)\right]}_{\text{heat kernel for } S^1} dy$$

32 If you have a function f on $[0,1]$, $f(0)=0=f(1)$

reflect by oddness $f(-x) = -f(x)$ on $[-1,1]$

periodic period 2 on $\mathbb{R}$.

no connection between $f(0)$ and $f(1)$

vs. S^1 f on $[0,1]$

periodic with period 1

$f(0) = f(1)$

$f^{(k)}(0) = f^{(k)}(1)$

3//

$$g(x) = \int_0^\infty \Phi_H(x,t)\, dt. = a\,\Phi_L(x) + b$$

$$= \left(\int_0^1 + \int_1^\infty\right) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}\, dt.$$

as $t \to 0$ $\quad -\frac{x^2}{4t} \to -\infty \qquad \Phi_H(x,t) \to 0 \qquad x \neq 0$

so $\int_0^1 \Phi_H\, dt$ is finite

$$\int_1^\infty \Phi_H\, dt \le \int_1^\infty \frac{1}{(4\pi t)^{n/2}} = \text{finite} \quad n \geq 3$$

$(\Delta - \partial_t)\Phi_H = 0$

$$\Delta g = \int_0^\infty \Delta \Phi_H\, dt = \int_0^\infty \partial_t \Phi_H\, dt$$

$$= \Phi_H(x,\infty) - \Phi(x,0)$$

$$= 0 - 0$$

$$= 0$$

In general $\quad \Delta u = \partial_t u \quad$ and apply Laplace transform in t $\quad v(x,\lambda) := \int_0^\infty u(x,t)\, e^{-\lambda t}\, dt$

$$\Delta v = \lambda v + u(x,0)$$

$$\Delta \tilde{\Phi}_H = \lambda \tilde{\Phi}_H + 0$$

For 1D $\quad (x,t) \in \mathbb{R} \times \mathbb{R}^+$

$$v_{xx} = \lambda v + h(x) \quad \text{an ODE with parameter.}$$