8b) $\int_C x \, d\sigma$ $\qquad \int f \, d\sigma = \int f\circ\Phi \sqrt{\det[(\Phi')^T\Phi]} \, dx$

$\Phi : t \mapsto (\overset{x}{\cos t}, \overset{y}{\sin t}) \qquad t\in(0,2\pi)$

$\Phi : \mathbb{R} \to \mathbb{R}^2$

$\Phi' = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

$\Phi' : \mathbb{R} \to \mathbb{R}^2$

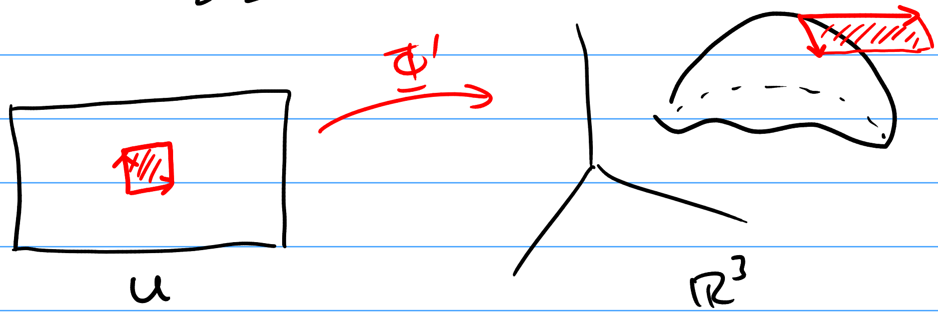
$\Phi'[v] = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$\int_C x \, d\sigma = \int_0^{2\pi} \cos t \times 1 \, dt$$

$$= \Big[\sin t\Big]_0^{2\pi} = 0 - 0 = 0$$

$$(\Phi')^T\Phi' = \begin{bmatrix} -\sin t & \cos t \end{bmatrix}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

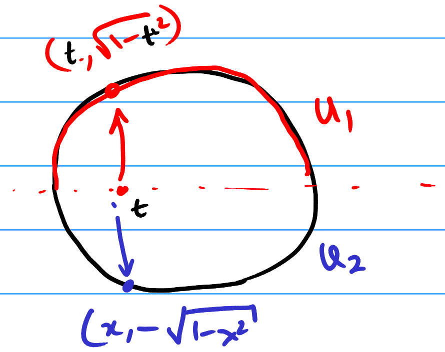
$$= [1]$$



If Φ is just a change of coordinates then $\Phi : \mathbb{R}^k \to \mathbb{R}^k$

$$\sqrt{\det[(\Phi')^T\Phi']} = \sqrt{(\det \Phi')^2} = |\det \Phi'|$$

c)



$$\Phi_1(t) = \left(\begin{matrix} t \\ \sqrt{1-t^2} \end{matrix}\right)$$

$$\Phi'(x) = \left(\begin{matrix} 1 \\ \frac{-x}{\sqrt{1-x^2}} \end{matrix}\right)$$

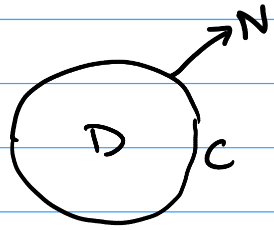
$$(\Phi')^T\Phi' = 1^2 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1 + \frac{x^2}{1-x^2}$$

$$= \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$\int_{u_1} x \, d\sigma = \int_{-1}^{1} t \times \frac{1}{\sqrt{1-t^2}} \, dt = \Big[-\sqrt{1-t^2}\Big]_{t=-1}^{t=1} = 0.$$

$$\int_{u_2} x \, d\sigma \ldots\ldots = 0.$$

(e) Compute $\int_C x\, d\sigma$ using Div Thm.



$C = \partial D$

$\int_{\partial\Omega} \underline{F \cdot N}\, d\sigma = \int_\Omega \nabla F\, dx$

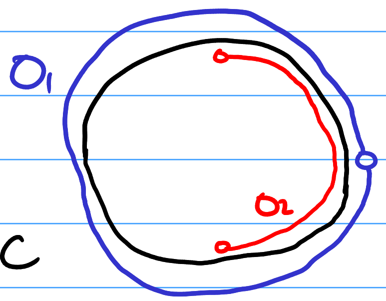
$\int_{\partial D} x\, d\sigma$

$N = (x-a, y-b)\frac{1}{r} = (x, y)$ $\qquad F = (F_1, F_2)$

$F \cdot N = F_1 x + F_2 y = x\ ?$ $\qquad F = \begin{pmatrix}1\\0\end{pmatrix}$

$\int_C x\, d\sigma = \int_{\partial D} \begin{pmatrix}1\\0\end{pmatrix} \cdot N\, d\sigma = \int_D \nabla \begin{pmatrix}1\\0\end{pmatrix} dx = \int_D 0\, dx = 0$

d) Partition of Unity



$O_1 = C \setminus \{(1,0)\}$

$O_2 = \{(x,y) \in C \mid x > 0\}$

$C = O_1 \cup O_2$

$\Phi_1 : (0, 2\pi) \to O_1$

$t \mapsto (\cos t, \sin t)$

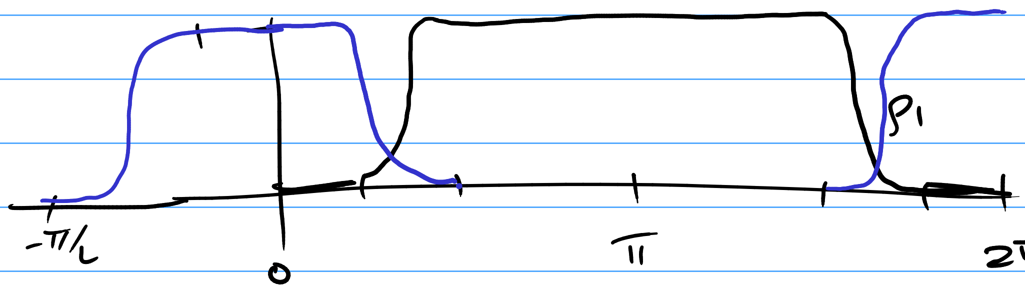
$\Phi_2 : (-\pi/2, \pi/2) \to O_2$

$t \mapsto (\cos t, \sin t)$

Choose a smooth $h : (0, 2\pi) \to [0,1]$

so that $h \equiv 1$ on $[\pi/2, 3\pi/2]$

$h \equiv 0$ for $t \le \pi/4$ or $t \ge 7\pi/4$

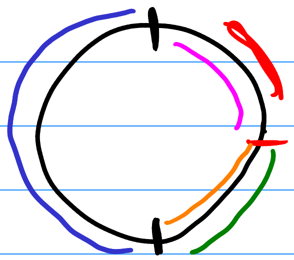


$\rho_1 = h \circ \Phi_1^{-1}$ $\qquad \underline{\rho_2} = 1 - \rho_1$ $\qquad (\rho_1 + \rho_2)(q) = 1$ for all $q \in C$.

$$\int_C f\, d\sigma = \sum_{p=1,2} \int_{\Phi_i^{-1}[O_i]} \rho_i \circ \Phi_i \times f \circ \Phi_i \times \sqrt{\det \Phi'^T \Phi'}\; dt$$

$$= \int_0^{2\pi} h(t) \times \cos t \times 1\, dt + \int_{-\pi/2}^{\pi/2} (1-h(t)) \times \cos t \times 1\, dt$$

$$= \left(\underline{\int_0^{\pi/2}} + \underline{\int_{\pi/2}^{3\pi/2}} + \underline{\int_{3\pi/2}^{2\pi}}\right) h(t) \cos t\, dt + \left(\underline{\int_{-\pi/2}^{0}} + \underline{\int_0^{\pi/2}}\right) (1-h(t)) \mathrm{cot}\, dt$$



$$= \int_{\pi/2}^{3\pi/2} \cos t\, dt + \int_0^{\pi/2} \cancel{h(t) \cos t} + (1 - \cancel{h(t)}) \cos t\, dt$$

2π $\quad 1-h(\cancel{t})$ $\quad$ cot

$$+ \int_{3\pi/2}^{2\pi} \cancel{h(t) \cos t}\, dt + \int_{-\pi/2}^{0} (1-h(t))\cos t\, dt$$

$3\pi/2$ $\qquad t' = t + 2\pi$

$$= \int_0^{2\pi} \cos t\, dt = 0$$

10. $B_r = \{x^2+y^2 \le r^2\}$ $\quad g(x,y) = \ln(x^2+y^2)$ $\quad r^2 = x^2+y^2$ on ∂B_r

a) $\displaystyle\int_{\partial B_r} \nabla g \cdot N \, d\sigma$ $\qquad \nabla g = \frac{2}{r^2}(x,y)$

$\qquad N = \frac{1}{r}(x,y)$

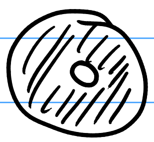
$$= \int_{\partial B_r} \frac{2}{r^3}(x,y)\cdot(x,y)\, d\sigma$$

$$= \int_{\partial B_r} \frac{2}{r}\, d\sigma \qquad r(x,y) = \sqrt{x^2+y^2}$$

$$= \frac{2}{r}\int_{\partial B_r} d\sigma = \frac{2}{r}\times 2\pi r = 4\pi.$$

b) Can't use Div thm directly ☹. $\quad f = \nabla g$

$\displaystyle\int_{B_r \setminus \{(0,0)\}} \nabla\cdot\nabla g \, dx$ $\qquad \nabla g$ is cts and diff on $B_r \setminus \{(0,0)\}$



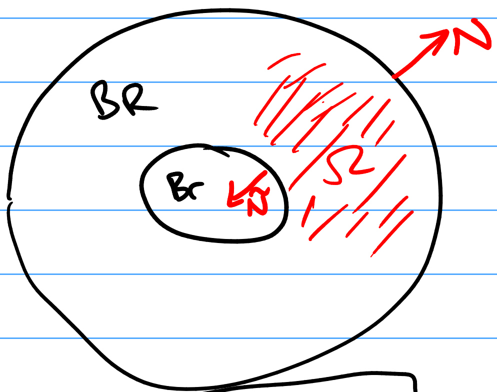
$$\partial(B_r \setminus \{(0,0)\}) = \underbrace{\partial B_r}_{\text{submanifold dim 1}} \cup \underbrace{\{(0,0)\}}_{\text{submanifold of dim 0.}}$$

$r < R$ $\qquad \Omega = B_R \setminus B_r$

$\qquad \partial\Omega = \partial B_R \cup \partial B_r$

$$\int_{B_R\setminus B_r} \nabla\cdot\nabla g \, dx = \int_{\partial B_R} \nabla g\cdot N\, d\sigma + \int_{\partial B_r} \nabla g \cdot \tilde{N}\, d\sigma$$

$$= \int_{\partial B_R} \nabla g\cdot N\, d\sigma - \int_{\partial B_r} \nabla g\cdot N\, d\sigma$$



$\tilde{N} = -N$

Yes, check $\boxed{\nabla\cdot\nabla g = 0}$ (when $(x,y) \ne (0,0)$)

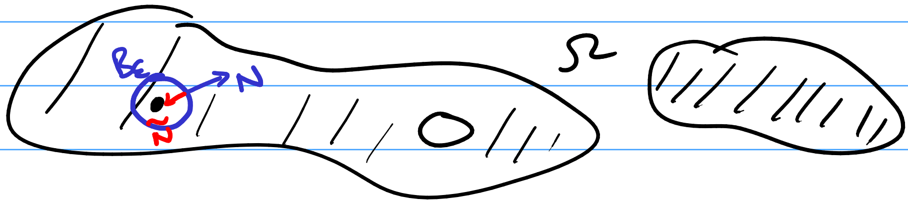
$$\int_\Omega \nabla\cdot\nabla g\, dx = \int_\Omega 0 \, dx = 0$$

Laplacian

c) If $(0,0) \notin \Omega$ then $\nabla \cdot \nabla g = \Delta g = 0$

$$\int_{\partial\Omega} \nabla g \cdot N \, d\sigma = \int_{\Omega} 0 = 0$$

If $(0,0) \in \Omega$ then Ω is open so $\exists\, \varepsilon$ $B_\varepsilon \subset \Omega$



$\tilde{\Omega} = \Omega \setminus B_\varepsilon$

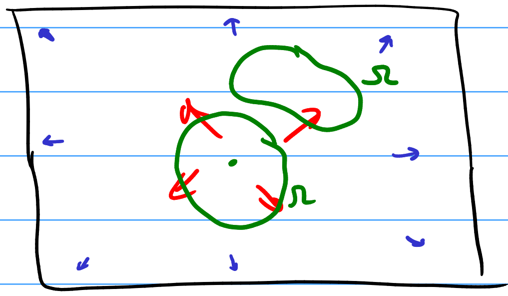
$$\int_{\tilde{\Omega}} \nabla \cdot \nabla g \, dx = \int_{\partial\Omega} \nabla g \cdot N \, d\sigma + \int_{\partial B_\varepsilon} \nabla g \cdot \tilde{N} \, d\sigma$$

$$= \int_{\partial\Omega} \nabla g \cdot N \, d\sigma - \int_{\partial B_\varepsilon} \nabla g \cdot N \, d\sigma$$

$$0 = \int_{\partial\Omega} \nabla g \cdot N \, d\sigma - 4\pi$$

d). ∇g as a flow

$\int_S F \cdot N \, d\sigma =$ " Flux of F " across S



11b) commutivity: $f * g = g * f$

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

let $z = x - y$

$z = \Phi(y) = x - y$

$\Phi' = -I$

$|\det \Phi'| = 1$

$$= \int_{\mathbb{R}^n} f(x-z)\, g(z)\, 1\, dz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y) g(x-y)\, dy_1 dy_2 \cdots dy_n$$

$z_1 = x_1 - y_1$

$z_2 = x_2 - y_2$

$\vdots$

$$= \int_{\infty}^{-\infty} \cdots\cdots \int_{\infty}^{-\infty} f(x-z) g(z)\, (-dz_1)(-dz_2) \cdots (-dz_n)$$

$$= \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} f(x-z) g(z)\, dz_1 \cdots dz_n$$

$$\int_a^b f(x)\, dx = \int_{[a,b]} f(x)\, dx$$