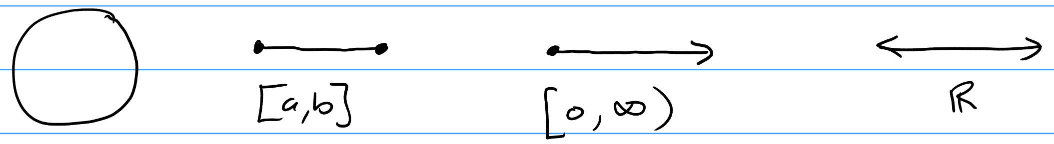
Tutorial 11

All closed 1-dimensional manifolds (with boundary)



$[a,b]$ $\quad$ $[0,\infty)$ $\quad$ $\mathbb{R}$

$\underbrace{\mathbb{R}/c\mathbb{Z}}_{33}$ $\quad$ $\underbrace{}_{36}$ $\quad$ $\underbrace{\phantom{\mathbb{R}}}_{\text{lecture}}$

$\mathbb{R}/\mathbb{Z}$ ✓ $\quad$ $[0,1]$ ✓ $\quad$ ✓ $\quad$ ✓

33. One method

write down a complete system of eigenfunctions

$H_{[0,b]}(x,y,t)$ $\qquad$ $[0,b] \longrightarrow [0,1]$

$$x \longmapsto \frac{1}{b}x$$

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

$\uparrow$ $n/2$ $\quad n=1$

$[0,b]$ $\qquad$ $[0,1]$

$$\frac{1}{b}\Phi\left(\frac{1}{b}x, \frac{1}{b^2}t\right) = \frac{1}{b} b \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) = \Phi(x,t)$$

$$H_{[0,b]}(x,y,t) = \frac{1}{b} H_{[0,1]}\left(\frac{1}{b}x, \frac{1}{b}y, \frac{1}{b^2}t\right)$$

$$H_{[0,b]} - \Phi(x-y,t) = \frac{1}{b} H_{[0,1]}\left(\frac{1}{b}x, \frac{1}{b}y, \frac{1}{b^2}t\right) - \frac{1}{b}\Phi\left(\frac{1}{b}(x-y), \frac{1}{b^2}t\right)$$

$$= \frac{1}{b}\left\{ H_{[0,1]}(x',y',t') - \Phi(x'-y',t') \right\}$$

has (ii)

$$H_{[a,b]}(x,y,t) := H_{[0,b-a]}(x-a,\, y-a,\, t)$$

$$H_{[a,b]}(x,y,t) - \Phi(x-y,t) = H_{[a,b]}(x,y,t) - \Phi((x-a)-(y-a),t)$$

3b.a) $u \in C^2(\overline{\Omega} \times (0,\infty))$ $\quad \Omega = (0,\infty)$, $u(0,t) = 0$

$$\tilde{u} = \begin{cases} u & x>0 \\ -u(-x,t) & x<0 \\ 0 & x=0 \end{cases} \quad \text{clearly also } C^2$$

Well it's cts. $\displaystyle\lim_{x\to 0^-} \tilde{u}(x,t) = \lim_{x\to 0^-} -u(-x,t) = -\lim_{y\to 0^+} u(y,t) = 0$ $\quad x=-y$

$$(\partial_t u)(0,t) = \lim_{h\to 0} \frac{u(0,t+h) - u(0,t)}{h} = \lim \frac{0-0}{h} = 0$$

$\partial_t u \in C^1(\overline{\Omega} \times (0,\infty))$

$$\partial_t \tilde{u} = \begin{cases} \partial_t u & x>0 \\ -\partial_t u(-x,t) & x<0 \\ 0 & x=0 \end{cases}$$

$\partial_{tt}\tilde{u}$ same argument.

$$\partial_x(-u(-x,t)) = (\partial_x u)(-x,t)$$

x-derivative

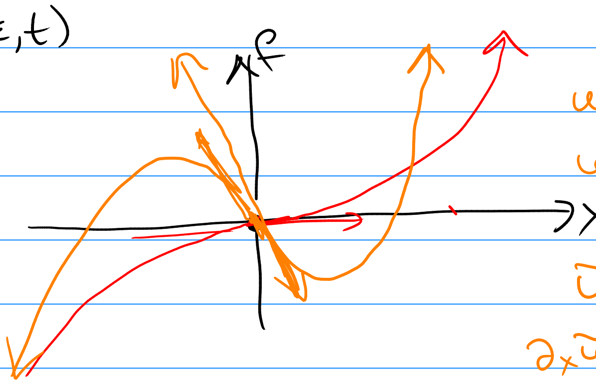
$$\partial_x \tilde{u} = \begin{cases} \partial_x u(x,t) & x>0 \\ \partial_x u(-x,t) & x<0 \\ \lim_{\varepsilon\to 0} \partial_x u(\varepsilon,t) \end{cases}$$

$\partial_x u \in C^1(\overline{\Omega} \times (0,\infty))$

$\Rightarrow$ extends continuously to $x=0$

Also

$\partial_x \partial_t \tilde{u}$ is cts on $\mathbb{R}$



$u(x)$ $x^2 - x = x(x-1)$

$u'(x) = 2x-1$

$u'' = 2$

$\tilde{u} = -x^2 - x$ for $x<0$

$\partial_x \tilde{u} = -2x-1$

$\partial_{xx}\tilde{u} = -2$

Only need to check $\partial_{xx}\tilde{u}$

$\partial_{xx} u$ is cts as $x \to 0^+$ $\qquad \partial_{xx} u = \partial_t u$ on $x \neq 0$

cts on $\mathbb{R}$

$y' = y$ if we show y is cts. and diff'ble

$\Rightarrow$ y is C^1

$\Rightarrow$ y is C^2

$\vdots$

$\Rightarrow$ y is C^∞

c) If u is a sol^n to HE on $[0,\infty)$ we know by (b) $\tilde{u}$ is a sol^n on $\mathbb{R}$.

and

$|\tilde{u}|$ is bounded

So maximum principle applies $|\tilde{u}|$ is bounded by $\sup|\tilde{h}|$
$= \sup|h|$

d) $$\tilde{u} = \int_{-\infty}^{\infty} \Phi(x-y,t)\, \tilde{h}(y)\, dy$$

$$= \int_{-\infty}^{0} \text{''} \, (-h(-y))\, dy + \int_0^\infty \text{''}$$

$= \text{end result.}$ $\int_0^\infty$ (funda sol^n + coord trans for fund sol^n) $h(y)\, dy$.

Harmonic half plane $G(x,y) = \Phi(x-y) - \Phi(x-R(y))$

34. $$\frac{1}{7}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$\tau = 7t$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0$$

35. $\dot{u} - \partial_{xx} u = 0 = f \quad \bar{\Omega} = [0,\pi]$

$u(0,t) = u(\pi,t) = 0 = g$

$u(x,0) = h(x) = x^2(\pi - x)$

$$u(x,t) = \int_0^\pi H_{[0,\pi]}(x,y,t)\, h(y)\, dy.$$

$$= \int_0^\pi \frac{1}{\pi} H_{[0,1]}\left(\frac{1}{\pi}x, \frac{1}{\pi}y, \frac{1}{\pi^2}t\right) h(y)\, dy$$

$$= \int_0^\pi \frac{2}{\pi} \sum_{k=1}^\infty e^{-k^2 t} \sin(kx) \sin(ky)\, h(y)\, dy$$

$$= \frac{2}{\cancel{\pi}} \sum \underbrace{e^{-k^2 t} \sin(kx)}_{\text{eigenfunction}} \underbrace{\int_0^\pi \sin(ky)\, y^2(\pi - y)\, dy}_{a_k}.$$

$$= -2\cancel{\pi} k^{-3}(1 + 2\underbrace{\cos k\pi}_{\pm 1})$$

k odd $\to -1$, k even $\to 3$.

$$= 4 \sum_{k \text{ odd}} k^{-3} e^{-k^2 t} \sin(kx)$$

$$- 12 \sum_{k \text{ even}} k^{-3} e^{-k^2 t} \sin(kx)$$