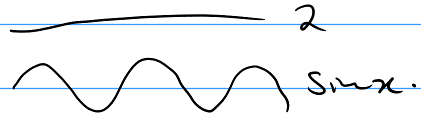
Tutorial 8

26. Is hard because the induction step is not clear

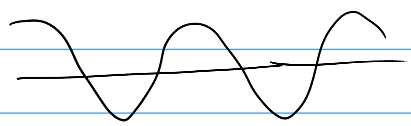
assume $\left|\partial^\alpha |x|^{-n}\right| \le A\, |x|^{-n-|\alpha|}$

$\left|\partial_i \partial^\alpha |x^{-n}|\right| \le A\, \partial_i |x|^{-n-|\alpha|}$ does not follow.

$\sin x \le 2$



$\underset{\cos x}{\underset{\shortparallel}{\partial \sin x}} \not\le \underset{0}{\underset{\shortparallel}{\partial 2}}$



$$\partial_k \frac{\text{poly}}{|x|^n} = \frac{\widetilde{\text{poly}}}{|x|^{n+1}}$$

b) $u(x) = \dfrac{r^2 - |\cancel{x} - x|^2}{n r \omega_n} \displaystyle\int_{\partial B(x,r)} \frac{u(y)}{|x-y|^n}\, d\sigma(y)$

$\partial(u(0)) = 0$ $\quad a = x$

$\partial u(x) = \dfrac{r}{n\omega_n}\, \partial_{x_i} \displaystyle\int_{\partial B(x,r)} \frac{u(y)}{|x-y|^n}\, d\sigma(y)$

$$\partial_{x_i} \int_{\partial B(0,1)} \frac{g(x+ry)}{\cancel{|y|^n}\ 1}\, d\sigma(y)$$

Green's Functions vs Fundamental Solution

$\Delta_x G(x,s) = \delta(x-s)$ and G is zero on boundary. unique, symmetric

A solution to $Lu = \delta$ on $\mathbb{R}^n$ don't care about boundary condition $\Delta\Phi(x) = \delta(x)$

$G - \Phi$ must be harmonic

$\Delta_x(G-\Phi) \overset{=0}{\Rightarrow} \Delta_x G = \Delta_x \Phi = \delta$

square root

Find the inverse of $\sqrt[+]{\ }$

$f: (0,1] \cup (-1,-2) \to (0,4)$

$x \longmapsto x^2$

$$f^{-1}(y) = \begin{cases} \sqrt[+]{y} & \text{for } y \in (0,1] \\ -\sqrt[+]{y} & \text{for } y \in (1,4) \end{cases}$$

24(b) Green's function for Δ on $H^+ = \{x_n > 0\}$

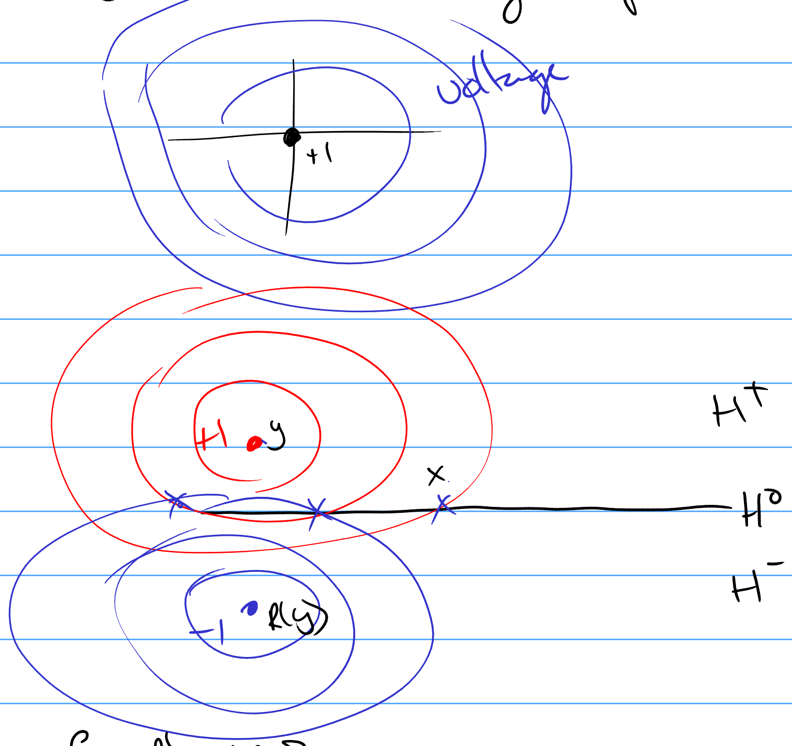
$$G_{H^+}(x,y) = \Phi(x-y) - \Phi(R(x)-y)$$
$$= \Phi(x-y) - \Phi(x - R(y))$$

So the voltage function satisfies $\Delta V =$ density charges

$G(x,y)$ = the voltage function at x which is zero on the boundary coming from a particle at y with electrical charge $+1$.

$\Delta_x G(x,y) = \delta(x-y)$ $\qquad \int_{\mathbb{R}^n} \delta(x-y) = +1$

$\Phi(x-y)$ is the voltage of a particle with charge $+1$ at y



for all $x \in \Omega$

i) $G(x,y) - \Phi(x-y)$ should be harmonic at all $y \in \Omega$

$G(x,y)$ is not defined for $x=y$.

Need to extend $G(x,y) - \Phi(x-y)$ to $x=y$.

ii) for all $x \in \Omega$ $\quad y \mapsto G(x,y)$ extends continuous to $\partial\Omega$ and be zero there

$$\tilde{G} = \begin{cases} G & \text{on } \Omega \\ 0 & \text{on } \partial\Omega \end{cases}$$

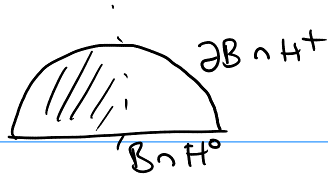
$$G_{H^+}(x) - \Phi(x-y) = \Phi(\underbrace{\overset{H^-}{R(x)} - \overset{H^+}{y}}_{}) = \frac{\text{Const}}{|R(x)-y|^{n-2}}$$

is never zero on $x, y \in H^+$

For $y \in H^0 = \partial H^+$ $\qquad R(y) = y$

$$G(x,y) = \Phi(x-y) - \Phi(x-R(y)) = 0.$$

c) Greens function on B^+



$$G_{B^+}(x,y) = G_B(x,y) - G_B(R(x),y)$$
$$= G_B(x,y) - G_B(x,R(y))$$

So if $y \in H^0$ $y = R(y)$

$$G_{B^+}(x,y) = 0$$

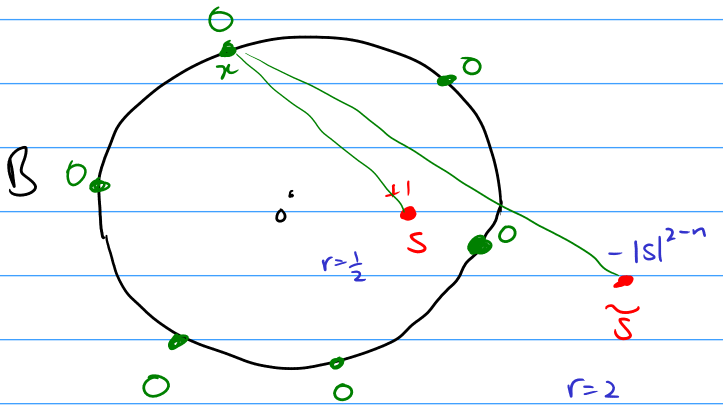
We already know if $y \in \partial B$ that $G_B(x,y) = 0$

$$G_{B^+}(x,y) = 0 - 0 = 0$$

$$G_{B^+}(x,y) - \Phi(x-y) = \underbrace{\left[G_B(x,y) - \Phi(x-y)\right]}_{\text{harmonic in } y.} - G_B(R(x),y)$$

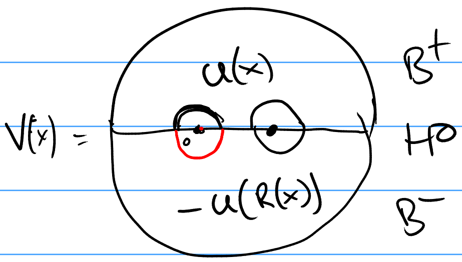
$R(x) - y \neq 0$

$$G_B(x,s) = V(x-s) - \underline{\underline{|s|^{2-n}}}\, V(x - \tilde{s}) \qquad n \geq 3$$



$\frac{1}{|x-s|^n}$

$|x-a| = c|x-b|$ in $\mathbb{R}^n$ is a sphere.



V is harmonic at x

$\iff$

for all small balls $B(x,r)$

$$V(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} V(y)\, d\sigma(y)$$

$$\frac{\partial^2}{\partial x_i^2} u(R(x)) = \left(\frac{\partial^2 u}{\partial x_i^2}\right)(R(x)) \qquad i \neq n$$

$$\frac{\partial^2}{\partial x_n^2}\left(u(x_1, \dots, x_{n-1}, -x_n)\right) = \cancel{(-1)}^2\left(\frac{\partial^2}{\partial x_n^2} u\right)(R(x)).$$

$$\int_{\partial B(x,r)} V(y)\, d\sigma(y) = \int_{\partial B^+} u(y)\, d\sigma + \int_{\partial B^-} -u(R(y))\, d\sigma(y) \qquad z = R(y)$$

$$= \int_{\partial B^+} u(y)\, d\sigma - \int_{\partial B^+} u(z)\, d\sigma(z) = 0 = V(x)$$

Alternate

$g = V|_{\partial B}$ this is continuous on ∂B

We know there is a unique soln $\tilde{V}$ to $\Delta \tilde{V} = 0$ on B and $\tilde{V}|_{\partial B} = g$.

By calculation $-\tilde{V}(R(y))$ is also a sol$^{\underline{n}}$

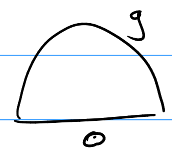
$\tilde{V}(y) = -\tilde{V}(R(y))$

$\Rightarrow \tilde{V} = 0$ on H^0

$\tilde{V} - u$ on B^+

$\Delta(\tilde{V} - u) = 0$

$\tilde{V} = u$ on ∂B^+



$\tilde{V} - u = 0$ by maximum principle

$\tilde{V} = V$ on B^+

$\Rightarrow$ on B too.