Scheduling the exam: In the 3rd week of Jan
or in the 1st, 2nd, 3rd of Feb

Feb

Mo 8

Di 9

Mi 10 X

Do 11 X

Fr 12

Google form to choose times.

Repeat at end of Feb.

$$\partial_x \int_a^y f(x,t)\,dt =$$

17.

$$\lim_{r\to 0} F(r) = \lim \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} f(y)\, d\sigma(y) \qquad y = x_0 + rz$$

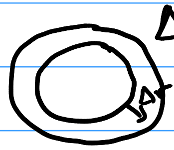
$$= \lim \frac{1}{n\omega_n \cancel{r^{n-1}}} \int_{\partial B(0,1)} f(x_0+rz)\, \cancel{r^{n-1}}\, d\sigma(z)$$

$$= \frac{1}{n\omega_n} \int_{\partial B(0,1)} \lim_{r\to 0} f(x_0+rz)\, d\sigma(z)$$

$$= \frac{1}{n\omega_n} \int_{\partial B(0,1)} f(x_0)\, d\sigma(z) = \frac{f(x_0)}{n\omega_n}\, n\omega_n = f(x_0).$$

$$\int_{B(0,1)} dx = \omega_n, \text{ in } \mathbb{R}^n \qquad \mathrm{vol}(B(0,r)) = \omega_n r^n$$

$$\sigma(\partial B(0,r)) = \underline{n\omega_n}\, r^{n-1}$$



$$\Delta V = \Delta r \times S$$

$$S = \frac{\Delta V}{\Delta r}$$

$$0 \le |F(r) - f(x_0)| = \left| \frac{1}{\sigma} \int_{\partial B} f(x)\, d\sigma - f(x_0) \right|$$

$$= \left| \frac{1}{\sigma} \int_{\partial B} f(x)\, d\sigma - \frac{1}{\sigma} \int_{\partial B} f(x_0)\, d\sigma \right|$$

$$= \left| \frac{1}{\sigma} \int (f(x) - f(x_0))\, d\sigma \right|$$

$$\le \frac{1}{\sigma} \int |f(x) - f(x_0)|\, d\sigma$$

$$\le \frac{1}{\sigma} \int \operatorname*{ess\,sup}_{\sigma} |f(x) - f(x_0)|\, d\sigma$$

$$\le \sup | \quad | \times 1$$

$$\to 0. \text{ as } r \to 0.$$

The role of the mean value theorem

$u \in C^2(\bar{\Omega})$

$$
\begin{tikzcd}
\begin{matrix} \Delta u = 0 \\ \text{"harmonic functions"} \end{matrix} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "3.3"] & \begin{matrix} \text{"harmonic distribution } U\text{"} \\ \Delta U = 0 \\ \text{ie } U(\Delta\varphi) = 0 \ \forall \varphi \text{ testfunctions} \end{matrix} \arrow[d, Rightarrow, "3.6"] \\
\begin{matrix} u \text{ is a mean value} \\ \text{function} \end{matrix} & \text{weak mean value property} \arrow[ul, Rightarrow, "U = F_u"', "\text{Weyl's Lemma}"]
\end{tikzcd}
$$

more general

How to prove $Lu = 0$ has a solution? L an elliptic operator

1. Prove there exists a weak solution, ie a distribution $LF = 0$

2. Prove the distribution comes from a function $F = F_u$

If u is harmonic ie $\Delta u = 0$ and $V = F_u$

$$V(\varphi) = \int u \varphi$$

$$(\Delta V)(\varphi) = \sum_k (\partial_k \partial_k V)(\varphi) = \sum_k -(\partial_k V)(\partial_k \varphi) = \sum V(\partial_k^2 \varphi)$$

$$= V(\Delta \varphi)$$

$$= \int u(\Delta \varphi) = \sum \int u(\partial_k \partial_k \varphi) = \sum_k - \int \partial_k u \, \partial_k \varphi$$

$$= \sum \int \partial_k^2 u \, \varphi = \int \Delta u \cdot \varphi = 0$$

$$\Delta(F_u) = F_{(\Delta u)}$$

18a) How do you calculate the Laplacian in $\mathbb{R}^2$ in polar coordinates?

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$r^2 = x^2 + y^2 \qquad \varphi = \arctan\frac{y}{x}$$

$$\frac{\partial}{\partial x} = \underline{\frac{\partial r}{\partial x}}\frac{\partial}{\partial r} + \underline{\frac{\partial \varphi}{\partial x}}\frac{\partial}{\partial \varphi}$$

$$= \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}$$

$$2r\frac{\partial r}{\partial x} = 2x = 2r\cos\varphi$$

$$\frac{\partial r}{\partial x} = \cos\varphi$$

$$\frac{\partial\varphi}{\partial x} = \frac{1}{\left(\frac{y}{x}\right)^2+1} \times -\frac{y}{x^2}$$
$$= \frac{-y}{x^2+y^2} = -\frac{r\sin\varphi}{r^2}$$
$$= -\frac{\sin\varphi}{r}$$

$$\frac{\partial}{\partial y} = \sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

$$\nabla u\cdot N = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot(\cos\varphi, \sin\varphi)$$
$$= \cos\varphi\,\partial_x u + \sin\varphi\,\partial_y u$$

$$\int_{\partial\Omega} F\cdot N = \int_\Omega \nabla\cdot F$$

$$\int_{\partial B}\nabla u\cdot N = \int_B \nabla\cdot\nabla u = \int_B \Delta u = 0$$

$$\frac{\partial u}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial u}{\partial y} = \cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}$$

$$\int_{\partial B}\frac{\partial u}{\partial r}\,d\sigma = \int_{\partial B}\nabla u\cdot N =$$

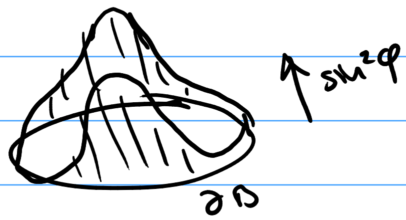
Neumann problem

b ii) Is is a harmonic function, $\Delta u = 0$ on B, and on ∂B $\frac{\partial u}{\partial r} = \underline{\underline{\sin^2\varphi}}_{n}$?

Poisson eqn

$\Delta u = f$ on Ω, $u|_{\partial\Omega} = g$. has a solution

$\int_{\partial B} \sin^2\varphi \neq 0.$



i) Suppose $u(r,\varphi) = f(r)\sin\varphi$ with $f'(1) = 1$

$$\Delta u = f'' \sin\varphi + \frac{1}{r} f' \sin\varphi + \frac{1}{r^2} f(-\sin\varphi) = 0$$

$$r^2 f'' + r f' - f = 0 \qquad \text{DE} \qquad f = r^k \Rightarrow k = \pm 1$$

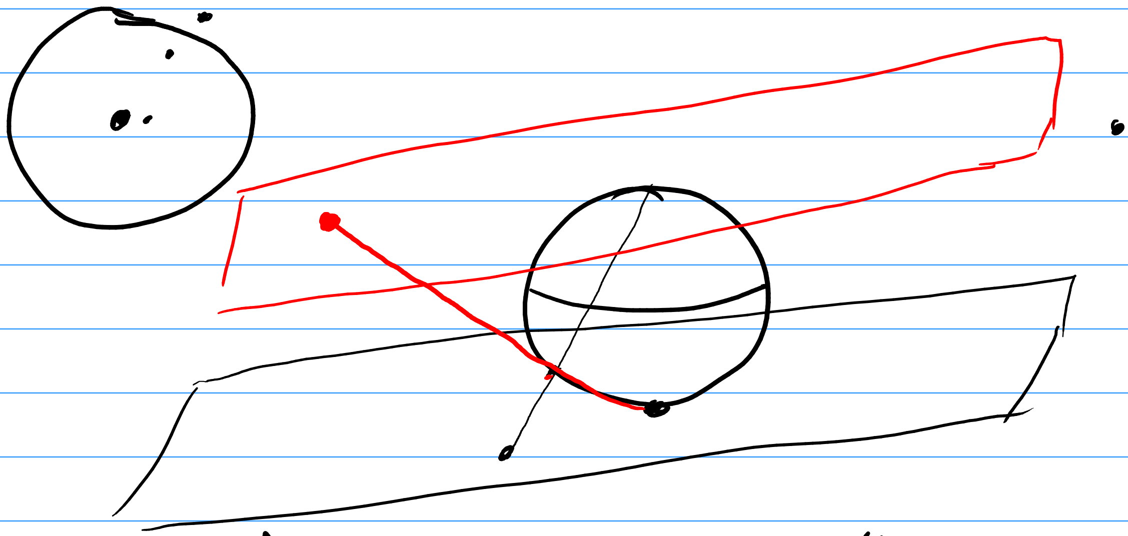
$$f = A r^{-1} + B r$$

$u \in C^2(\overline{B})$ so $A = 0$

$$f(r) = Br \qquad f'(r) = B \qquad f'(1) = B = 1$$

$$u(r,\varphi) = r\sin\varphi$$

$u(x,y) = y$ $\Leftarrow$ "of course" this is harmonic



"Spherical harmonics"

20a) $$F(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} v(x+rz)\, d\sigma(z)$$

$$\frac{\partial}{\partial r} F(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \frac{\partial}{\partial r}\left(v(x+rz)\right) d\sigma(z)$$

$$= \frac{1}{n\omega_n} \int_{\partial B(0,1)} (\nabla v)(x+rz) \cdot N \, d\sigma(z)$$

$$= \frac{1}{n\omega_n} \int_{B(0,1)} \Delta v \, dz \geq 0$$

$$v(x) = \lim_{r \to 0} F(r) \leq F(r)$$

b) Assume there is a maximum $v(x_0)$ for $x_0 \in \Omega$ try to prove that v is constant.

$$\underset{\shortparallel}{v(x_0)} \leq \frac{1}{n\omega_n} \int_{\partial B(x_0,r)} v(y)\, d\sigma(y)$$

$$\frac{1}{n\omega_n} \int_{\partial B(x_0,r)} v(x_0)\, d\sigma(y)$$

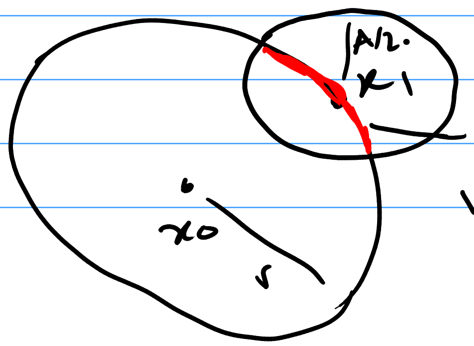
$$0 \geq \frac{1}{n\omega_n} \int_{\partial B(x_0,r)} \left(v(x_0) - v(y)\right) d\sigma(y)$$

Suppose there is a point $x_1 \in \partial B(x_0,r)$ which is less

$v(x_0) > v(x_1)$ $\quad v(x_0) - v(x_1) = A > 0$

By continuity $\forall \varepsilon > 0\ \exists \delta > 0 \quad |x - x_1| < \delta \Rightarrow |v(x) - v(x_1)| < \frac{A}{2}$

$B(x_1, \delta) \cap \partial B(x_0, r)$ is non empty. it has some σ-volume / surface area M



$v(x_1) - \frac{A}{2} \leq v(x) \leq v(x_1) + \frac{A}{2}$

$$\int_{\text{red area}} v(x_0)-v(x)\ d\sigma(x) \geq \int_{\text{red}} v(x_0)-v(x_1)-\frac{A}{2}$$

$$= \int_{\text{red}} A-\frac{A}{2} = \frac{A}{2}\int_{\text{red}} = \frac{A}{2}M > 0$$

$$0 \geq \int_{\partial B} v(x_0)-v(x) = \underbrace{\int_{\text{red}} v(x_0)-v(x)}_{\geq \frac{A}{2}M} + \underbrace{\int_{\text{nonred}} v(x_0)-v(x)}_{\geq 0}$$

$$\geq \frac{A}{2}M > 0$$

The contradiction.

$$r^2 f'' + r f' - f = 0$$

Singular analysis of ODES.

$$f = r^k = \exp(k \ln r)$$

$$f = \exp g.$$
$$f' = g' \exp g$$
$$f'' = g'' \exp g + (g')^2 \exp g$$

$$\frac{d}{dr}\left(r^n f'\right) = n r^{n-1} f' + r^n f''$$

$$r^2 f'' + 2 r f' - r f' - f = 0$$
$$(r^2 f')' - (r f)' = 0$$
$$r^2 f' - r f = A$$
$$r f' - f = A/r$$