1a) $\partial^{(0,2,1)} f = \partial^0_{x_1} \partial^2_{x_2} \partial^1_{x_3} f = \frac{\partial^2}{(\partial x_2)^2} \frac{\partial}{\partial x_3} f.$

the zeroth derivative is just the function

$= \partial_2^2 \partial_3 f.$

b) $\partial_2 \partial_1 \qquad \partial_1 \partial_2 \qquad \partial^{(1,1,0)}$

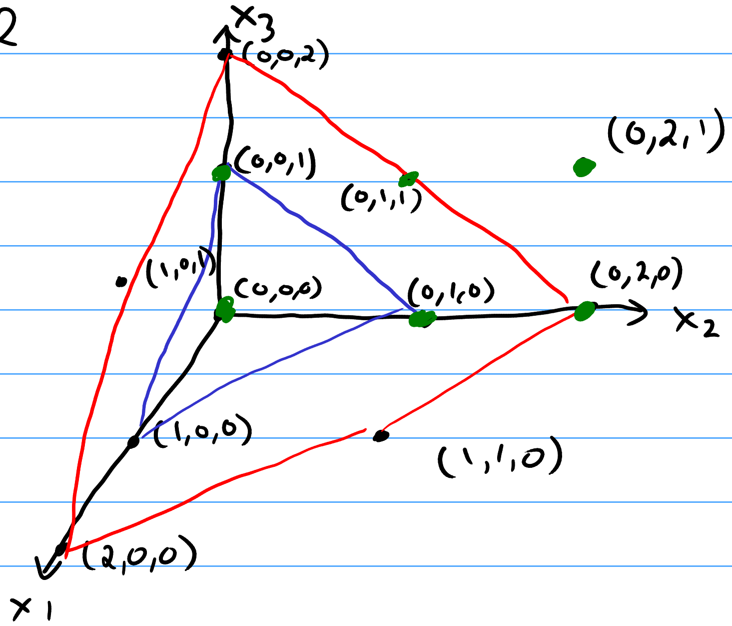
c) $\delta \le \gamma \iff 0 \le \gamma - \delta \iff$ every component of $\gamma - \delta \geq 0$

$(1,2,0) \le (1,3,0) \qquad (0,1)$ and $(1,0)$

$(1,2,0) \not\le (1,1,1)$

$|(1,2,0)| = 1+2+0 = 3$

$|\gamma| \le 2 \qquad \gamma \le (0,2,1)$



$\binom{(0,2,1)}{(0,1,0)} = \binom{0}{0}\binom{2}{1}\binom{1}{0} = 1 \times 2 \times 1 = 2.$

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n}$$

$$= \left[\binom{\gamma_1-1}{\delta_1-1} + \binom{\gamma_1-1}{\delta_1}\right]\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n}$$

$$= \binom{\gamma_1-1}{\delta_1-1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n} + \binom{\gamma_1-1}{\delta_1}\cdots\binom{\gamma_n}{\delta_n}$$

$$= \binom{\gamma-e_1}{\delta-e_1} + \binom{\gamma-e_1}{\delta}$$

e) $\gamma = 0$

$$\sum_{0\le\delta\le 0}\binom{0}{\delta}\partial^{\delta}u\,\partial^{0-\delta}v = \binom{0}{0}uv = uv = \partial^0(uv)$$

Assume formula is true for $|\gamma| = k$. Prove it for $|\gamma'| = k+1$

$\gamma' = \gamma + e_j$ we will prove it for $e_j = e_1 = (1, 0, \ldots)$

$$\partial^{\gamma+e_1}(uv) = \partial_1\partial^{\gamma}(uv) = \partial_1\left[\sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^{\delta}u\,\partial^{\gamma-\delta}v\right]$$

$$= \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^{\overbrace{\delta+e_1}^{\delta'}}u\,\partial^{\gamma-\delta}v + \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^{\delta}u\,\partial^{\gamma'-\delta}v$$

$\delta' = \delta + e_1$

$$= \sum_{e_1\le\delta'\le\gamma'}\binom{\gamma}{\delta'-e_1}\partial^{\delta'}u\,\partial^{\gamma'-\delta'}v + \text{" "}$$

$$= \binom{\gamma}{\gamma}\partial^{\gamma'}u\,\partial^{0}v + \sum_{e_1\le\delta\le\gamma}\underbrace{\left[\binom{\gamma}{\delta-e_1} + \binom{\gamma}{\delta}\right]}_{\binom{\gamma'}{\delta}}\partial^{\delta}u\,\partial^{\gamma'-\delta}v$$

$$+ \sum_{\substack{0\le\delta\le\gamma \\ \delta_1 = 0}}\binom{\gamma}{\delta}\partial^{\delta}u\,\partial^{\gamma'-\delta}v$$

$$= \binom{\gamma'}{\gamma'} \partial^{\gamma'} u \, \partial^{0} v \qquad + \sum_{\substack{0 \le \delta \le \gamma \\ \delta_1 \ge 1}} \binom{\gamma'}{\delta} \partial^{\delta} u \, \partial^{\gamma' - \delta} v$$

$$+ \sum_{\substack{0 \le \delta \le \gamma \\ \delta_1 = 0}} \binom{\gamma'}{\delta} \partial^{\delta} u \, \partial^{\gamma' - \delta} v$$

$\delta_1 = 0$ $\qquad \binom{\gamma}{\delta} = \binom{\gamma + e_1}{\delta} = \binom{\gamma'}{\delta}$

$\qquad \qquad \qquad \shortparallel$

$\binom{\gamma_1 + 1}{0} \binom{\gamma_2}{\delta_2} \cdots \binom{\gamma_n}{\delta_n}$

$$= \binom{\gamma'}{\gamma'} \partial^{\gamma'} u \, \partial^{0} v \quad + \sum_{0 \le \delta \le \gamma} \binom{\gamma'}{\delta} \partial^{\delta} u \, \partial^{\gamma' - \delta} v$$

$$= \sum_{0 \le \delta \le \gamma'} \binom{\gamma'}{\delta} \partial^{\delta} u \, \partial^{\gamma' - \delta} v$$

$0 \le \delta \le \gamma' = \gamma + e_1$

either $\quad \delta \le \gamma \quad$ or $\quad \delta = \gamma + e_1$

2. $\int \frac{du}{dt} = \int t^2$

$$u = \frac{1}{3}t^2 + c.$$

a) $\frac{du}{dt} = f(t)$

$$\int_0^t u' \, ds = \int_0^t f \, ds$$

$$u(t) - u(0) = \int_0^t f(s)\, ds$$

$$u(t) = u(0) + \int_0^t f(s)\, ds.$$

b) $$\underline{u(t) = c + \int_0^t f(s)\, ds}$$

$$u(0) = c + \cancel{\int_0^0} = c.$$

If there is another solution v, the u-v solves the homogeneous.

$$\frac{d}{dt}(u-v) = \frac{du}{dt} - \frac{dv}{dt} = f - f = 0$$

$$u - v = \text{const.} = 0$$
$$u(0) - v(0) = c - c = 0$$

The homogeneous initial value problem has a unique sol^n.

$$u : \Omega \to \mathbb{R}$$

$$\nabla \cdot u = \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \vdots \\ \partial_n u \end{pmatrix} \in \Omega \to \mathbb{R}^n$$

$u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ $\qquad$ $\cancel{\mathbb{R}^{n+1}}$ $\qquad$ $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

c) $$v(x,t) = \int_0^t f(\underbrace{x + (s-t)b}_{\mathbb{R}^n}, \underbrace{s}_{\mathbb{R}})\, ds$$

$$\partial_t v = f(x + (t-t)b, t) + \int_0^t \frac{d}{dt} f(x+(s-t)b, s)\, ds$$

$$= f(x,t) + \int_0^t \nabla f(x+(s-t)b, s) \cdot \underbrace{\frac{d}{dt}(x + (s-t)b)}_{-b}\, ds$$

$$= f(x,t) - b \cdot \int_0^t \nabla f(x+(s-t)b, s)\, ds .$$

$$\nabla v = \int_0^t \nabla f(x+(s-t)b, s)\, ds$$

$$\partial_t v + b \cdot \nabla v = f(x,t) - b \cdot \int + b \cdot \int = f(x,t).$$

$$v(x,0) = \int_0^0 = 0 .$$

$$\underline{u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\, ds}$$

$$u(x,0) = g(x) + \int_0^0 = g(x)$$

$$\partial_t g(x-tb) = \nabla g(x-tb) \cdot (-b)$$

$$\partial_t g(x-tb) + b \cdot \nabla g(x-tb) = 0$$