6a) $b \cdot \nabla u + cu = 0$

$$z(s) = u(\underset{\sim}{x}(s))$$

$$\frac{dz}{ds} = \nabla u \cdot \frac{dx}{ds} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial s}$$

We choose the path | characteristic $\dot{x} = b$

$$\begin{cases} \frac{dz}{ds} = \nabla u \cdot \dot{x} = \nabla u \cdot b = -c(x\ \ )z \\ \frac{dx}{ds} = b(x) \end{cases}$$

Goal of solving a PDE is to find $u(x)$ ie find z.

b) $\dot{p}$ = something

$$\begin{cases} \dot{z} = \text{only } z \text{ and } x \\ \dot{x} = \text{only } z \text{ and } x \end{cases}$$

$$F = b(x,z) \cdot p + c(x,z) = 0$$

example with $n=1$ $\quad x(s) : \mathbb{R} \to \mathbb{R}$

$F = b(x,z)\, p + c(x,z)$ $\qquad z = u(x) \quad p = \frac{dz}{dx} = \frac{du}{dx}$

$z(s) = u(x(s))$

eg $F = uu' - \frac{1}{2}x^2 = 0 \longrightarrow (u')^2 + uu'' - x = 0$

$F = zp - \frac{1}{2}x^2 = 0$ $\qquad \underline{\underline{uu''}} = x + p^2$

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} \qquad\qquad \frac{dp}{ds} = \frac{d}{ds}\left(\frac{du}{dx}\right) = \frac{d^2u}{dx^2}\frac{dx}{ds}$$

$\boxed{\dot{u} = p\dot{x}}$ always $\qquad \boxed{\dot{x} = u}$ choice $\qquad = u''\dot{x} = x + p^2$

$\dot{u} = pu$

Hess(u) $\qquad$ not have higher derivatives

$$\frac{\partial}{\partial x_1} F(\overset{p}{u'}, \overset{z}{u}, \overset{x}{x}) = 0$$

$$\underline{\underline{\frac{\partial F}{\partial p}}} \; \frac{\partial u'}{\partial x_1} + \dots = 0$$

↑ second derivative

c) $\partial_t u + b \nabla u = f(x)$ $\qquad$ $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (x,t)

$$\tilde{\nabla} u = \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \vdots \\ \partial_n u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} \nabla u \\ \partial_t u \end{pmatrix} \qquad \tilde{x} = (x,t)$$

$$\begin{pmatrix} b \\ 1 \end{pmatrix} \cdot \tilde{\nabla} u - f(x) = 0$$

$$\begin{pmatrix} b \\ 1 \end{pmatrix} \cdot p - f(x) = 0 \qquad \text{vs} \qquad \overset{(b)}{b \cdot p - c = 0}$$

$$\dot{\tilde{x}} = \begin{pmatrix} b \\ 1 \end{pmatrix} \qquad \dot{z} = -c = f(x)$$

$$\tilde{x} = \begin{pmatrix} b \\ 1 \end{pmatrix} s + \tilde{x}_0 \qquad \int_0^t \dot{z}\, ds = \int_0^t f(x(s))\, ds$$

$$t = s + t_0 \qquad z(t) - z(0) = \int_0^t f(bs + x_0)\, ds$$

$$t(s) = s$$

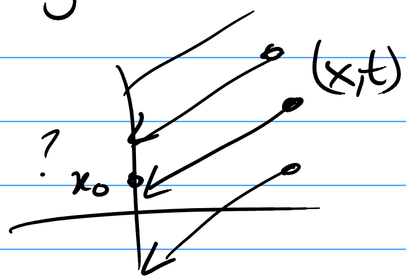
$$z(t) = z(0) + \int_0^t f(bs + \underbrace{x(s) - bt(s)}_{x - bt})\, ds$$

$$= g(x - bt) + \dots$$

$$x(s) = bs + x_0$$

$$x(s) = b\, t(s) + x_0$$

$$x_0 = x(s) - bt(s)$$



7 a) $x_1 \partial_1 u + x_2 \partial_2 u = 2u \qquad x_1 \in \mathbb{R} \quad x_2 > 0 \qquad u(x_1, 1) = g(x_1)$

$$F(p,z,x) = x_1 p_1 + x_2 p_2 - 2z = 0$$

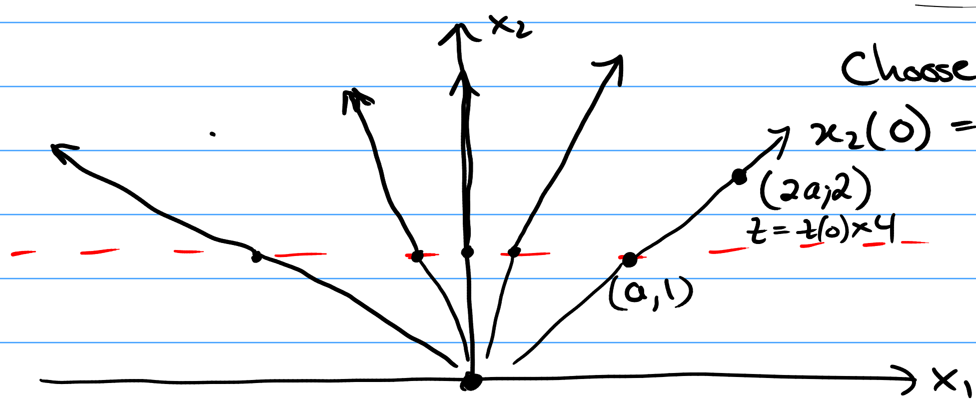
$$= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - 2z$$

$$b \cdot p + c$$

6(b) $\Rightarrow \quad \dot{z} = -c = 2z \qquad\qquad z = z(0)e^{2s} = z(0)(e^s)^2$

$\dot{x} = b \Rightarrow \quad \dot{x}_1 = x_1 \quad \Rightarrow \quad x_1 = a\ e^s$

$\dot{x}_2 = x_2 \qquad\qquad x_2 = 1\ e^s$



Choose $s=0$ so that $x_2(0) = 1$, ie is the initial point.

$1 = be^0$

$b = 1$

$be^{s+s_0} = (be^{s_0})e^s$

$s = 0 \qquad x(0) = (a,1)$

$x(s) = e^s(a,1)$

$z(s) = z(0)e^{2s}$

$x_1(s) = a e^s$

$x_2(s) = e^s$

$z(s) = u(x_1(0), x_2(0))\,(x_2(s))^2$

$= u(a,1)\ x_2^2$

$= g(a)\, x_2^2$

$x_1 = a x_2$

$a = \frac{x_1}{x_2}$

$z(s) = g\left(\frac{x_1(s)}{x_2(s)}\right)(x_2(s))^2$

$$\boxed{u(x_1,x_2) = g\left(\frac{x_1}{x_2}\right) x_2^2}$$

b) $x_1 \partial_2 u - x_2 \partial_1 u - u = 0 \qquad x_1, x_2 > 0 \qquad u(x_1, 0) = g(x_1)$

$x_1 p_2 - x_2 p_1 - z = 0$

$\binom{-x_2}{x_1} \cdot p - z = 0$

$$6(b) \Rightarrow \begin{cases} \mathring{z} = z \\ \dot{x}_1 = -x_2 \\ \mathring{x}_2 = x_1 \end{cases}$$

$z = c\, e^s$

$\ddot{x}_1 = -\dot{x}_2 = -x_1$

$\ddot{x}_1 = -x_1$

$x_1 = a \cos s + b \sin s$

$x_2 = a \sin s - b \cos s$

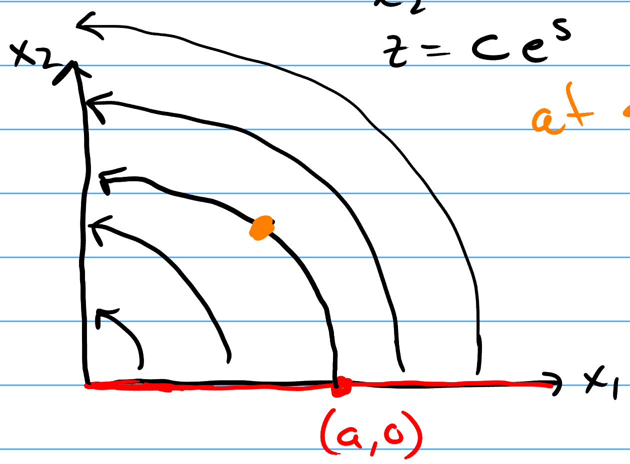
Freedom to choose the condition for $s=0$. $\quad x_2(0) = 0$

$0 = 0 - b \times 1 \quad \Rightarrow b = 0$

$x_1 = a \cos s$

$x_2 = a \sin s.$

$z = c e^s$



at • $\left(\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}a\right)$ ie $s = \frac{\pi}{4}$

$z = c e^{\pi/4}$

$a = \sqrt{x_1^2 + x_2^2}$

$\frac{x_2}{x_1} = \tan s$

$s = \arctan\left(\frac{x_2}{x_1}\right)$

$$z(0) = c = u(x_1(0), x_2(0)) = u(a,0) = g(a)$$
$$= g\left(\sqrt{x_1^2 + x_2^2}\right)$$

$$u(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}\right) e^{\arctan(x_2/x_1)}$$

$\mathring{z} = -c = 2z$ $\qquad \frac{dz}{ds} = 2z \quad \frac{dx_1}{ds} = x_1 \quad \frac{dx_2}{ds} = x_2$

$\mathring{x} = b \Rightarrow \begin{matrix} \dot{x}_1 = x_1 \\ \mathring{x}_2 = x_2 \end{matrix}$ $\qquad ds = \frac{dz}{2z} = \frac{dx_1}{x_1} = \frac{dx_2}{x_2}$

2 DEs : $\frac{dz}{dx_1} = \frac{2z}{x_1} \qquad \frac{dx_2}{dx_1} = \frac{x_2}{x_1}$

$\ln z = 2\ln x_1 + c \qquad \ln x_2 = \ln x_1 + a$

$z = C x_1^2 \qquad x_2 = A x_1$

$A = x_2/x_1$

initial condition $(x_1, x_2) = (y, 1)$ then $z = g(y)$

$1 = Ay \qquad y^2 C = g(y)$

$y = A^{-1} = \frac{x_1}{x_2} \qquad C = y^{-2} g(y)$

$= \left(\frac{x_2}{x_1}\right)^2 g\left(\frac{x_1}{x_2}\right)$

$z = Cx_1^2 = \left(\frac{x_2}{x_1}\right)^2 g\left(\frac{x_1}{x_2}\right) x_1^2$

$u(x_1, x_2) = x_2^2 g(x_1/x_2)$.

d) $u\partial_1 u + \partial_2 u - 1 = 0 \qquad x_1, x_2 > 0 \qquad u(y,y) = \frac{1}{2}y$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - 1 = 0$$

$$\dot{z} = 1 \Rightarrow z = s + c$$
$$\dot{x}_1 = z(s) \Rightarrow \dot{x}_1 = s + c \Rightarrow x_1 = \frac{1}{2}s^2 + cs + a$$
$$\dot{x}_2 = 1 \Rightarrow x_2 = s + a$$

$a = b$

We want $s=0$ to be in the initial value situation $x_1(0) = x_2(0)$

$$c = z(0) = u(x_1(0), x_2(0)) = u(a,a) = \frac{1}{2}a$$

$$z = s + \frac{1}{2}a$$
$$x_1 = \frac{1}{2}s^2 + \frac{1}{2}as + a$$
$$x_2 = s + a \qquad a = x_2 - s$$

$$x_1 = \cancel{\frac{1}{2}s^2} + \frac{1}{2}(x_2 - \cancel{s})s + x_2 - s$$
$$= \frac{1}{2}x_2 s + x_2 - s$$
$$x_1 - x_2 = \left(\frac{1}{2}x_2 - 1\right)s$$

$$s = 2\frac{x_1 - x_2}{x_2 - 2}$$

$$z = u(x_1, x_2) = \frac{1}{2}a + s = \frac{1}{2}(x_2 - s) + s = \frac{1}{2}x_2 + \frac{1}{2}s$$
$$= \frac{1}{2}x_2 + \frac{x_1 - x_2}{x_2 - 2}.$$