1D wave eqn $(\partial_{xx} - \partial_{tt})u = 0$

37 b) Want to show $u(x,t) = F(x+t) + G(x-t)$ is a solution

Transport eqn $(\partial_x - \partial_t)u = 0$

$\xi = x+t \qquad \eta = x-t$

$$\partial_x = \frac{\partial \xi}{\partial x}\partial_\xi + \frac{\partial \eta}{\partial x}\partial_\eta = \partial_\xi + \partial_\eta$$

$$\partial_t = \partial_\xi - \partial_\eta$$

$$\begin{aligned}\partial_x\partial_x - \partial_t\partial_t &= (\partial_\xi + \partial_\eta)(\partial_\xi + \partial_\eta) - (\partial_\xi - \partial_\eta)(\partial_\xi - \partial_\eta)\\ &= \cancel{\partial_\xi^2} + 2\partial_\xi\partial_\eta + \cancel{\partial_\eta^2} + (\cancel{\partial_\xi^2} + 2\partial_\eta\partial_\xi + \cancel{\partial_\eta^2})\\ &= 4\partial_\xi\partial_\eta\end{aligned}$$

Eg of more complicated expanding of brackets.

$$\begin{aligned}(\partial_\xi + \xi\partial_\eta)^2 &= (\partial_\xi + \xi\partial_\eta)(\partial_\xi + \xi\partial_\eta)u\\ &= (\partial_\xi + \xi\partial_\eta)(\partial_\xi u + \xi\cdot\partial_\eta u)\\ &= \partial_\xi^2 u + \underline{[\partial_\eta u + \xi\partial_\xi\partial_\eta u]} + \xi\partial_\eta\partial_\xi u + \xi^2\partial_\eta^2 u\end{aligned}$$

c) u solves 1D wave $\Leftrightarrow$ $u = F(\xi) + G(\eta)$

$u(x,t=0) = g(x)$, $\partial_t u(x,t=0) = h(x)$.

$t=0 \Leftrightarrow \xi = \eta = x$

$$\begin{cases} F(\xi) + G(\xi) = g(\xi) & (1)\\ F'(\xi) - G'(\xi) = h(\xi) & (2)\end{cases}$$

$$H(\xi) = \int_0^\xi h(\Xi)\,d\Xi.$$

$$F(\xi) - G(\xi) = H(\xi) + C \quad (2')$$

We can solve each part of the initial cond separately.

$$(\partial_{xx}-\partial_{tt})u=0$$

$u(x,0)=g(x) \qquad \partial_t u(x,0)=0 \qquad$ A

$u(x,0)=0 \qquad \partial_t u(x,0)=h(x). \qquad$ B

$F(\xi)+C(\xi)=g(\xi) \qquad \tilde{F}=F+K \quad \breve{C}=C-K$

$F(\xi)-C(\xi)=F(0) \qquad u=F(\xi)+C(\eta)$

$C(\xi)=F(0)+F(\xi) \qquad C(0)=0$

38

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0 \qquad \tau=at$$

$$\frac{\partial^2 u}{\partial \tau^2}-\frac{\partial^2 u}{\partial x^2}=0 \qquad t=0 \Leftrightarrow \tau=0$$

$$\frac{\partial u}{\partial t}=\frac{\partial \tau}{\partial t}\frac{\partial u}{\partial \tau}$$

$u(x,\tau=0)=g(x) \qquad \partial_t u(x,t=0)=h(x)$

$a\partial_\tau u=h(x)$

$\partial_\tau u(x,\tau=0)=\frac{1}{a}h(x).$

$$u(x,\tau)=\frac{1}{2}\left[g(x+\tau)+g(x-\tau)\right]+\frac{1}{2}\int_{x-\tau}^{x+\tau}\frac{1}{a}h(y)\,dy$$

$$=\frac{1}{2}\left[g(x+at)+g(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}h(y)\,dy$$

39 $u \in C^2(\bar{\Omega})$ $\quad U \subseteq \mathbb{R}^n$

$(\Delta - \partial_{tt}) u = 0$ is a classical solution

$u \in C^0(\bar{\Omega})$ is called a weak solution

$\int (\Delta \varphi - \partial_{tt} \varphi)\, u = 0$ for all test functions φ.

The distribution F associated to u. ie $F(\varphi) = \int \varphi u$

F solves wave eqn,

$((\Delta - \partial_{tt}) F)(\varphi) = 0$ as dist. $\forall \varphi$

$\Leftrightarrow$

$F(\Delta \varphi \qquad \partial_{tt} \varphi) = 0$

$\Leftrightarrow$

$\int (\Delta \varphi - \partial_{tt} \varphi)\, u = 0$

a) $\int_U \int_0^T \underline{(\partial_{tt} \varphi) u} \; - \underline{(\Delta \varphi) u}$

$\int_0^T (\partial_{tt} \varphi) u = \cancel{\partial_t \varphi u}\Big|_0^T - \int_0^T (\partial_t \varphi)(\partial_t u)$

$= - \cancel{\varphi \partial_t u}\Big|_0^T + \int_0^T \varphi\, \partial_{tt} u$

Green's thm.

$\int_U (\Delta \varphi)\, u = \cancel{\int_{\partial U} (u \nabla \varphi - \varphi \nabla u) \cdot N \, d\sigma}$

$+ \int_U \varphi (\Delta u)$

$\int_\Omega \underbrace{((\Delta - \partial_{tt}) \varphi)\, u}_{0} = \int_\Omega \varphi \underbrace{(\Delta - \partial_{tt}) u}_{0}$

u classical $\Rightarrow$ u weak

u weak sol^{n} $\Rightarrow$ $(\Delta - \partial_{tt}) u = 0$ ie classical sol^{n}.

$\forall \varphi$

b) $$\int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u_k = 0$$

$$\int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u$$

every point $p \in \Omega$ has a n'hood V_p we have $\sup_{u \in V_p} |u(y) - u_k(y)| \to 0$

0

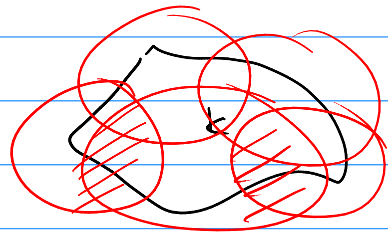
$$\left| \int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u - \lim_{k\to\infty} \int (\partial_{tt}\varphi - \Delta\varphi)\, u_k \right|$$

$$= \left| \lim_{k\to\infty} \left[\int (\partial_{tt}\varphi - \Delta\varphi)(u - u_k) \right] \right|$$

$$\leq \lim_{k\to\infty} \int_K |\partial_{tt}\varphi - \Delta\varphi|\, |u - u_k|$$

Notice φ has compact support $\subset K$ compact.

$$K \subset \bigcup_{p\in K} V_p \qquad K \subset \bigcup_{j=1,\dots,n} V_j$$



$$\leq \lim_{k\to\infty} \sum_{j=1}^n \int_{V_j} |\partial_{tt}\varphi - \Delta\varphi|\, |u - u_k|$$

$$\leq \lim \sum \sup_{V_j} |u - u_k| \underbrace{\int_{V_j} |\partial_{tt}\varphi - \Delta\varphi|}_{C_j}$$

$$= \sum_{j=1}^n C_j \lim_{k\to\infty} \sup_{V_j} |u - u_k|$$

$$= 0$$

40b) $$u_n = \sum_{k=1}^n (a_k \cos kt + b_k \sin kt) \sin kx$$

$$u = \lim_{n\to\infty} u_n$$

$$(\partial_{tt}-\partial_{xx})\cos kt \sin kx = -k^2 \cos kt \sin kx - (-k^2 \cos kt \sin kx)$$
$$= 0.$$

the terms u_n are C^2 functions $\Rightarrow$ strong solutions

$\Rightarrow$ u_n are weak solution.

the sum is uniformly convergent.

for all x, t

$$\left| \sum^{\infty} (a_k \cos kt + b \sin kt) \sin kx \right|$$
$$\leq \sum^{\infty} \left(|a_k||\cos kt| + |b_k||\sin kt| \right) |\sin kx|$$
$$\leq \sum^{\infty} |a_k| + |b_k| < \infty$$

$\Rightarrow$ u is a weak solution.

eg triangle wave

$$x_{\text{triangle}} = \frac{8}{\pi^2} \sum (-1)^n n^{-2} \sin(2\pi n t)$$

$$0=(\partial_{tt}-\partial_{xx})u=(\partial_t-\partial_x)\underbrace{(\partial_t+\partial_x)}_{\in \ker(\partial_t-\partial_x)} \qquad C^2$$

$$\operatorname{img}(\partial_t+\partial_x) \qquad \ker(\partial_t-\partial_x)$$