Tutorial 9

27 a) Follow by calculation

$$\partial_t u_\lambda = (\partial_t u) \cdot \frac{\partial \lambda^2 t}{\partial t} = \lambda^2 \dot{u}$$

$$\partial_i u_\lambda = (\partial_i u) \cdot \frac{\partial \lambda x_i}{\partial x_i} = \lambda \partial_i u$$

$$\partial_i^2 u_\lambda = \lambda \cdot \lambda \cdot \partial_i^2 u$$

$$\Delta u_\lambda = \lambda^2 \Delta u \qquad (\partial_t - \Delta) u_\lambda = \lambda^2 (\dot{u} - \Delta u) = 0.$$

b) $v(x,t) = x \cdot \nabla u + 2t \dot{u}$

$$\partial_\lambda u_\lambda = \partial_\lambda \left(u(\lambda x, \lambda^2 t) \right) = \nabla u \cdot \frac{\partial(\lambda x)}{\partial \lambda} + \dot{u} \cdot \frac{\partial \lambda^2 t}{\partial t}$$

$$= x \cdot \nabla u + 2\lambda t \dot{u} \qquad \text{for } \lambda = 1$$

$$= v(x,t).$$

Why is $\partial_\lambda u_\lambda$ a sol^n to $\dot{u} - \Delta u = 0$?

$$\partial_\lambda u_\lambda = \lim_{h \to 0} \frac{u_{\lambda+h}(x,t) - u_\lambda(x,t)}{h}$$

$=: q_{\lambda,h}(x,t)$ solves the heat equation by linearity.

so the limit $\quad (\partial_t - \Delta) q_{\lambda,h} = 0 \quad \forall \lambda, h, x, t$

$$(\partial_t - \Delta) \underbrace{\lim q_{\lambda,h}}_{= \partial_\lambda u_\lambda} = 0$$

c) If we have a solution $u(x,t) = v(\underbrace{t^{-1}x^2}_{z})$

$\partial_t - \Delta$

$$z = t^{-1}x^2$$

$$\partial_x = (\partial_x z)\,\partial_z = 2t^{-1}x\,\partial_z \qquad \frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial x}$$

$$\partial_x^2 = 2t^{-1}\partial_z + 2t^{-1}x\,\partial_x\partial_z$$

$$= 2t^{-1}\partial_z + 2t^{-1}x(2t^{-1}x)\,\partial_z\partial_z$$

$$= 2t^{-1}\partial_z + 4t^{-2}x^2\,\partial_z^2$$

$$\partial_t = \partial_t z\;\partial_z = -t^{-2}x^2\,\partial_z$$

$$\partial_t - \partial_x^2 = -t^{-1}(t^{-1}x^2)\,\partial_z - t^{-1}(2\partial_z + 4t^{-1}x^2\partial_z^2)$$

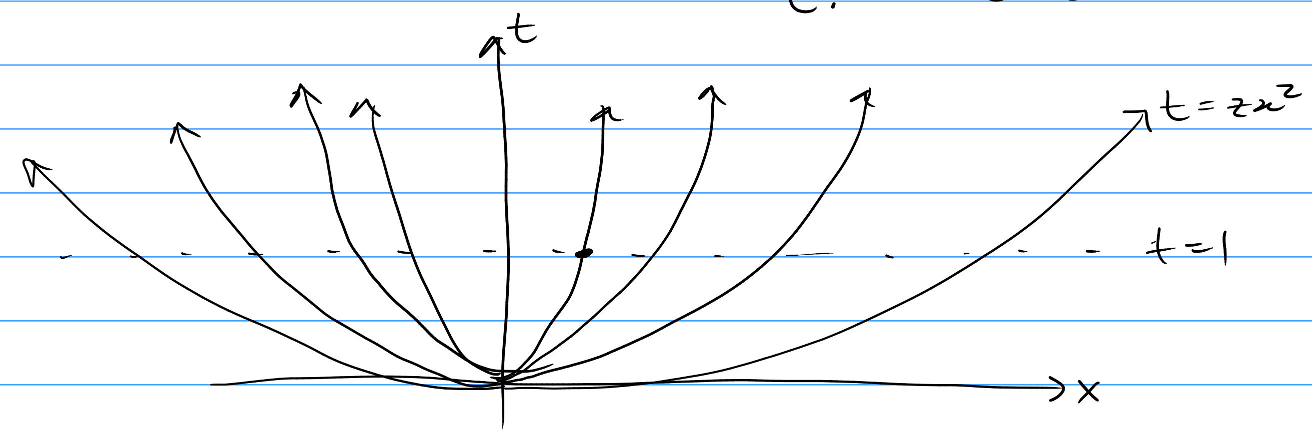
$$= -t^{-1}\left[z\,\partial_z + 2\partial_z + 4z\,\partial_z^2\right]$$

$$(\partial_t - \partial_x^2)\,u = -t^{-2}\left[\quad\right] v$$

Transport equation 1D. $\quad (\partial_t - \partial_x)\,F(x,t) = 0$

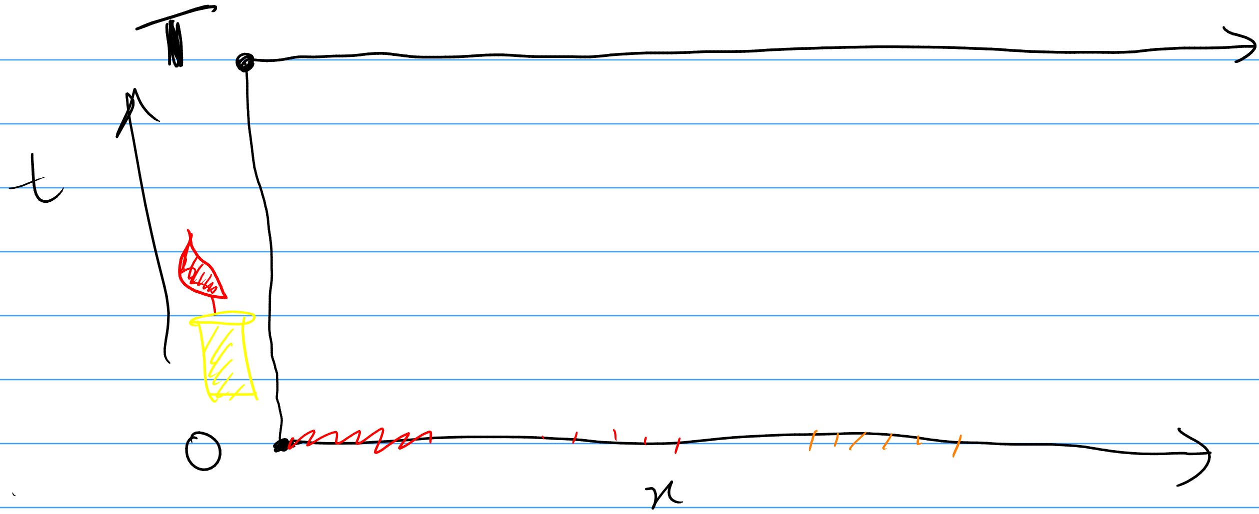
$\iff F(x,t) = G(x-t)$

"Characteristic" $\quad z = \frac{x^2}{t}, \quad t = zx^2$



Newton's Law of Cooling $\frac{\partial u}{\partial t} \sim$ difference

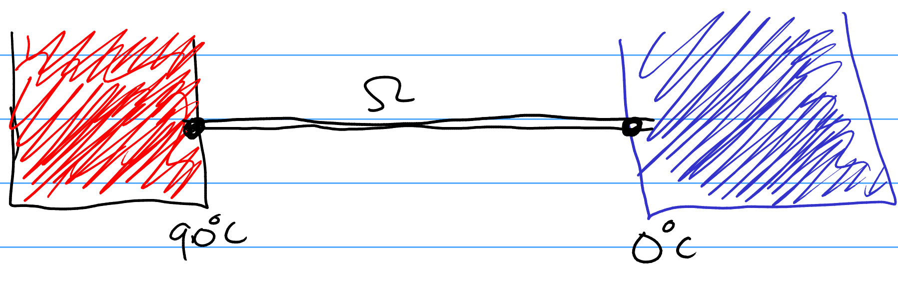
$$\partial_t u = \partial_x^2 u$$



$\Omega = [0, \infty)$ $\quad\Omega_T = \Omega \times (0, T)$

"Steady State Solution" = $\dot{u} = 0$

$0 - \Delta u = 0$ are harmonic function



Harmonic functions in 1D $\Rightarrow$ linear functions.

$\Omega = [0, 1]$ $\quad\partial\Omega = \{0, 1\}$

Black-Scholes PDE from Finance.

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b) Suppose that $u(x_0,t_0) = M$ is a maximum

$$M \le u(x_0,t_0) = \frac{1}{C}\int_{E(r)} u(y,s)\, \omega \, dy ds \le \frac{1}{C}\int_E M \omega \, dy ds$$

$$\parallel$$

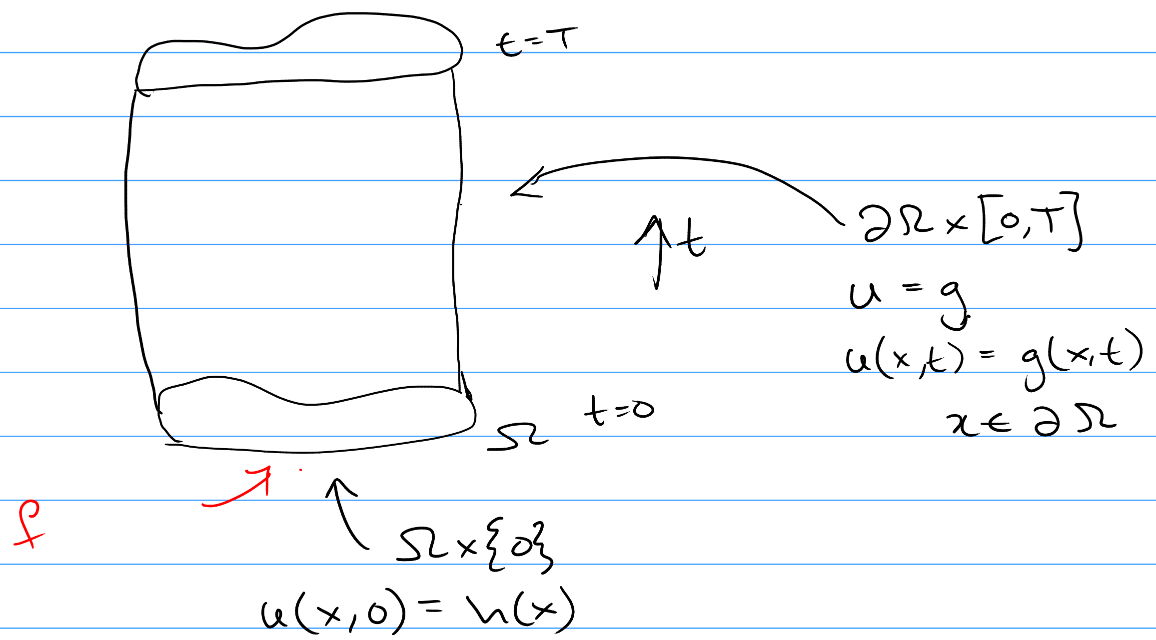
$$M$$

if $u(y_0,s_0) \neq M$ $\exists\, B(y_0,\varepsilon)$ where $u(x,t) \le M-\delta$

$$\frac{1}{C}\int_B u \, \omega \, dy ds \le (M-\delta) \times \frac{\text{weight volume ball}}{C}$$

$$< M$$

c)



$V = u_2 - u_1$ $\quad$ V sub-soln of heat eqn.

$$(\partial_t - \Delta) V = (\partial_t - \Delta) u_2 - (\partial_t - \Delta) u_1 = f_2 - f_1 \le 0$$

$V \le 0$ for all $(x,t) \in \Omega_T$

$u_2 \le u_1$

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$$\frac{du}{dt} - \Delta u = 0$$

$$\frac{du}{dt} = \Delta u$$

$$u = e^{\Delta t} f$$

$$u := \sum_0^\infty \frac{1}{k!} \Delta^k f \; t^k$$

$$\partial_t u = \sum_1^\infty \frac{1}{k!} k \, \Delta^k f t^{k-1} \qquad k-1 = m$$

$$= \sum_{m=0}^\infty \frac{1}{m!} (\Delta^{m+1} f) t^m$$

$$= \Delta \left(\sum_{m=0}^\infty \frac{1}{m!} \Delta^m f \; t^m \right)$$

$$= \Delta u.$$

$$u(x,0) = \sum_{k=0} \frac{1}{k!} (\Delta^k f)(x) \; 0^k = \frac{1}{0!} \cdot (\Delta^0 f)(x) \cdot 1 = f(x)$$

u is a well defined function by M-test.

$$\sum \left| \frac{1}{k!} \Delta^k f \; t^k \right| \leq \sum \frac{1}{k!} (Mt)^k = e^{Mt} < \infty$$

uniformly convergent on compact sets $A \subset \Omega \times \mathbb{R}$

Also shows infinitely differentiable in t.

$$\left| \partial_t^n a_k \right| = \left| \frac{k(k-1) \cdot \ldots \cdot (k-n+1)}{k!} (\Delta^k f) \, t^{k-n} \right| \leq \binom{k}{n} M^k t^{k-n}$$

Mean value thm of 1-D $\quad c \in (a,b)$

$$\left|(\partial_x f)(b) - (\partial_x f)(a)\right| \leq \left|\partial_x \partial_x f\,(c)\right| \left|b-a\right|$$

$$\partial_x^2 f\,(c) \quad \text{vs} \quad \Delta f$$

Not so certain this works for $\partial_x u$.