13(a) $F(\phi) = \int_{\mathbb{R}} x^3 \phi'' \, dx.$

- linear because multiplication, differentiation, integration are linear.

For any compact set $K \subset \mathbb{R}$ and $\phi \in C_0^\infty(K)$ that

$$|F(\phi)| = \left| \int_{\mathbb{R}} x^3 \phi'' \, dx \right| \leq \int_{\mathbb{R}} |x^3| \, |\phi''| \, dx$$

$$= \int_K |x^3| \, |\phi''| \, dx \leq \int_K |x^3| \, \sup_{x \in K} |\phi''(x)| \, dx$$

$$= \left(\underbrace{\int_K |x^3| \, dx}_{C} \right) \underbrace{\|\phi\|_{K,2}}_{\|\phi\|_{K,(2)}}$$

$$|F(\phi)| \leq C \, \|\phi\|_{K,2} \qquad \qquad \int f \phi$$

$$F(\phi) = \int_a^b x^3 \phi'' = \left[\cancel{x^3 \phi'} \right]_a^b - \int_a^b 3x^2 \phi' \, dx$$

$$= -\left[\cancel{3x^2 \phi} \right]_a^b + \int_a^b 6x \phi \, dx$$

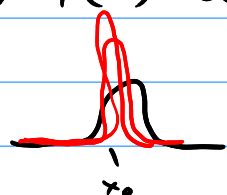$$= \int_{\mathbb{R}} \underbrace{6x}_{f(x)} \phi \, dx.$$
$$\qquad F = F_f \qquad f \in L^1_{loc}$$

(b) $|\delta(\phi)| = |\phi(0)| \leq \sup_{x \in c} |\phi(x)| = 1 \cdot \|\phi\|_{K,0}$

Why is $\delta \neq \int g \phi$

$G(\phi) = \int_{\mathbb{R}} g(x) \, \phi(x) \, dx$

Use mollifiers!

 $\{\phi_\varepsilon\}$ $\varepsilon > 0$ $\int \phi_\varepsilon = 1$

$\text{supp } \phi_\varepsilon \subseteq (x_0 - \varepsilon, x_0 + \varepsilon)$

$$G(\phi_\varepsilon) = \int_{\mathbb{R}} g(x)\, \phi_\varepsilon(x)\, dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} g\, \phi_\varepsilon\, dx$$

$$\left| G(\phi_\varepsilon) - g(x_0) \right| = \left| \int_{x_0-\varepsilon}^{x+\varepsilon} g\, \phi_\varepsilon\, dx - \int \frac{1}{2\varepsilon} g(x_0)\, dx \right|$$

$$\leq \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left| g\, \phi_\varepsilon - \frac{1}{2\varepsilon} g(x_0) \right| dx$$

$$\int \left(\min_{[x_0-\varepsilon,\, x_0+\varepsilon]} g \right) \phi_\varepsilon\, dx \leq G(\phi_\varepsilon) \leq \int \left(\max_{[x_0-\varepsilon,\, x_0+\varepsilon]} g \right) \phi_\varepsilon\, dx$$

$$\min g \leq G(\phi_\varepsilon) \leq \max g$$
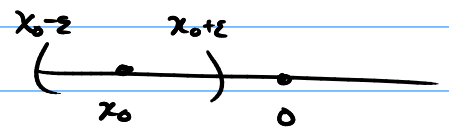
Take limit as $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \min_{[x_0-\varepsilon,\, x_0+\varepsilon]} g(x) = g(x_0)$$

$$\lim_{\varepsilon \to 0} G(\phi_\varepsilon) = g(x_0) \qquad \text{if } g \text{ is continuous}$$

"Lesbegue point" $\qquad f(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(y)\, dy.$

molifer ϕ_ε centered at $x_0 \neq 0$

$$\delta(\phi_\varepsilon) = \phi_\varepsilon(0) = 0 \quad \text{for} \quad \varepsilon \text{ small}$$

$$g(x_0) = 0 \quad \text{for all } x_0 \neq 0$$

$$G(\phi) = \int 0\, \phi = 0$$

(c) $\qquad F'(\phi) = -F(\phi') = -\int x^3 \phi''' \, dx \qquad F = F_{6x}$

$$= -\int 6x \phi'\, dx = \int \underline{6}\, \phi\, dx \qquad F' = F_6$$

$$L = \text{supp } \psi \qquad K \times L \text{ is compact}$$

14. Choose any compact $K \subset \mathbb{R}^m$ and any $\varphi \in C_0^\infty(K)$

$$|G(\varphi)| = |F(\varphi\psi)| \leq C_1 \|\varphi\psi\|_{K \times L, \alpha_1} + \cdots + C_j \|\varphi\psi\|_{K \times L, \alpha_j}$$

$$\|\varphi\psi\|_{K \times L, \alpha} = \sup_{(x,y) \in K \times L} |\partial^\alpha \varphi(x) \, \psi(y)| = \sup |\partial^{\alpha'} \varphi \, \partial^{\alpha''} \psi|$$

where $\quad \alpha = (\underbrace{\alpha'}_{n}, \underbrace{\alpha''}_{m})$

$$\leq \sup |\partial^\alpha \varphi| \, \sup |\partial^{\alpha''} \psi|$$

$$= \|\psi\|_{L, \alpha''} \, \|\varphi\|_{K, \alpha'}$$

$$|G(\varphi)| \leq \underbrace{C_1 \|\psi\|_{L, \alpha_1''}}_{\widetilde{C}_1} \|\varphi\|_{K, \alpha_1'} + \cdots$$
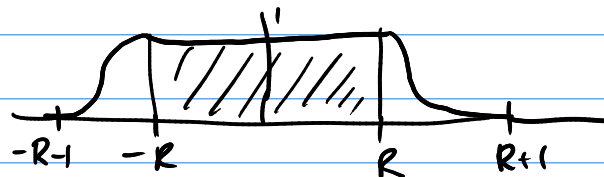
15.

$\mathbb{R}^2 = (x,t)$

$F(\phi) = \int_R \phi(0,t)\, dt \qquad \overset{\phi(x-x_0,t)}{F(\phi(x,t-t_0))} = F(\phi(x,t))$

To be continuous need to show $\forall K \subset \mathbb{R}^n$ and $\forall L \subset \mathbb{R}^{n+1}$
$\forall \phi \in C_0^\infty(L) \qquad\qquad I: C_0^\infty(\mathbb{R}^{n+1}) \rightarrow C_0^\infty(\mathbb{R}^n)$

$\|I(\phi)\|_{K,\alpha} \leq C_1 \|\phi\|_{L,\beta_1} + \dots + C_j \|\phi\|_{L,\beta_j}$

Why can't we just use $\|\cdot\|_\infty$ on $C_0^\infty(\mathbb{R}^n)$?

then $H(\phi) := \int_{\mathbb{R}^n} \phi\, dx$ would not be continuous

$\phi_R =$

$\text{supp } \phi_R \subset [-R-1, R+1]$

$\|\phi_R\|_\infty = 1$

$|H(\phi_R)| = \int_{-R-1}^{R+1} \phi_R\, dx \geq 2R$

H is not a bounded operator.

$F_f(\phi) = \int_{\mathbb{R}^n} f\, \phi\, dx \qquad f \in L_{loc}^1$

$\|I(\phi)\|_{K,\alpha} = \sup_K \left| \partial^\alpha \int_R \phi(x,t)\, dt \right|$

$D(\Omega) := C_0^\infty(\Omega, \mathbb{R})$ with the topology of family of seminorms

" test function"

$D'(\Omega)$ is distributions

$\sum_p A_i \|I(\phi)\|_{K,\alpha_i} \leq \sum_p \left(\sum_j C_{ij} \|\phi\|_{L,\beta_{ij}} \right)$

(b) $\varphi \in D(\mathbb{R}^n \times \mathbb{R})$ $\varphi \in \ker I \Leftrightarrow \exists \psi \in D(\mathbb{R}^n \times \mathbb{R})$
                                                     $\partial_t \psi = \varphi$

"$\Leftarrow$" $\exists \psi$ $\partial_t \psi = \varphi$ $[a,b] \supseteq \text{supp } \psi$

$$I(\varphi) = I(\partial_t \psi) = \int_a^b \partial_t \psi \; dt = \psi \Big|_a^b = 0 - 0 = 0$$
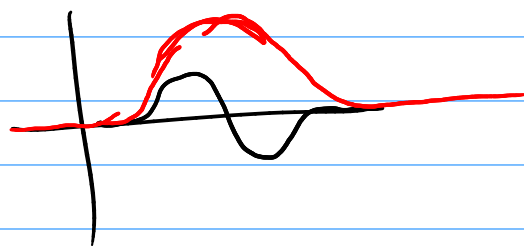
"$\Rightarrow$" If $\varphi \in \ker I$

$$\psi(x,t) := \int_{-\infty}^t \varphi(x,s) \, ds$$

$\exists T$ so that $\forall x \in \mathbb{R}^n$ $\forall t \notin [-T,T]$ $\varphi(x,t) = 0$

$t > T$ $\psi(x,t) = \int_{-\infty}^t \varphi = \int_{-\infty}^{-T} + \int_{-T}^{+T} + \int_{T}^t \varphi$

$$= 0 + \int_{-T}^{T} \varphi + 0$$

$$= 0 + 0 + 0 = 0$$

$$0 = \int_{\mathbb{R}} \varphi = \int_{-T}^{+T} \varphi$$

(c) $\partial_t F = 0 \Leftrightarrow F(\phi) = 0 \quad \forall \phi \in \ker I$

"$\Rightarrow$" $\partial_t F = 0$ Take any $\varphi \in \ker I$

$$F(\varphi) = F(\partial_t \psi) = -\partial_t F(\psi) = 0$$

"$\Leftarrow$" If $F(\phi) = 0 \quad \forall \phi \in \ker I$.   Take $\psi \in D$

$$\partial_t F(\psi) = -F(\underbrace{\partial_t \psi}_{\ker I}) = 0$$

(d) $\partial_t F = 0 \iff F(\phi) = C(I(\phi))$ Linear algebra question

$\updownarrow$

$F \equiv 0$ on $\ker I$

$$D(\mathbb{R}^n \times \mathbb{R}) \xrightarrow{\; I \;} D(\mathbb{R}^n)$$

$F \downarrow \qquad\qquad C$

$\mathbb{R} \quad \longleftarrow$

16

$$(\partial_t + b\partial_x)f = 0$$
$$(\partial_t + b\partial_x)F = 0$$

← derivatives of distributions

(a) $S : D \rightarrow D$

$$(S\phi)(x,t) = \phi(x-bt,t) \qquad , \partial_t , \partial_x$$

$$\partial_x(S\phi) = \partial_x(\phi(x-bt,t)) = (\partial_x\phi)(x-bt,t) = S(\partial_x\phi)$$

$$\partial_t(S\phi) = \partial_t(\phi(x-bt,t)) = \partial_x\phi(x-bt,t)\cdot -b + (\partial_t\phi)(x-bt,t)$$

$$= -b\, S(\partial_x\phi) + S(\partial_t\phi)$$

$$\widetilde{F}(\phi) := F(\tilde\phi) = F(S\phi) \qquad \color{red}{(\partial_t\widetilde{F})(\phi) \neq \partial_t F(\tilde\phi)}$$

$$(\partial_t\widetilde{F})(\phi) = -\widetilde{F}(\partial_t\phi) = -F(S(\partial_t\phi))$$

$$= -F(\partial_t(S\phi) + b\, S(\partial_x\phi))$$

$$= -F(\partial_t(S\phi)) - b\, F(\partial_x(S\phi))$$

$$= \partial_t F(S\phi) + b\, \partial_x F(S\phi)$$
$$= \underbrace{((\partial_t + b\partial_x)F)}_{O}(S\phi)$$

$$= 0$$

(e) we know   If F solves the Transport Eqn

then $\partial_t\widetilde{F} = 0$ part (a)

$$F(S(\phi)) = \widetilde{F}(\phi) = G(I(\phi))$$

$$F(\phi) = F(S\, S^{-1}\phi) = \widetilde{F}(S^{-1}\phi) = G(I(S^{-1}\phi))$$

part (d) $G\left(x \mapsto \int_{\mathbb{R}} \underset{\color{red}{S^{-1}\phi}}{\phi(x+bt,t)}\, dt\right) = \color{red}{G(I(S^{-1}\phi))}$