23. $f : \Omega \to \mathbb{R}$ $\quad g_k : \partial\Omega \to \mathbb{R}$

continuous.

$$-\Delta u = f \text{ on } \Omega, \quad u|_{\partial\Omega} = g_k$$

Suppose $g_1 \le g_2$ on $\partial\Omega$. Prove $u_1 \le u_2$

$V = u_2 - u_1$ $\quad$ Then $\Delta v = \Delta(u_2 - u_1) = -f - (-f) = 0$

And on $\partial\Omega$, $\quad v|_{\partial\Omega} = u_2|_{\partial\Omega} - u_1|_{\partial\Omega} = g_2 - g_1 \ge 0.$

What happens if there is a point $x \in \Omega$ so that

$u_1(x) > u_2(x)$

$\iff 0 > (u_2 - u_1)(x) = v(x).$

By the $\underbrace{\text{maximum principle}}_{\displaystyle \sup_{x\in\Omega} v \le \sup_{x\in\partial\Omega} v}$ (minimum principle) that $v \ge 0$.

$$v(x) \ge \inf_{x\in\Omega} v \ge \inf_{x\in\partial\Omega} v \ge 0 \qquad \forall x \in \Omega$$

$$u_2(x) - u_1(x) \ge 0$$

$$u_2 \ge u_1.$$

$\Delta u_1 = \Delta u_2 = 0$ and $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$ then $u_1 \equiv u_2$.

A holomorphic function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$

and

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(x+h) - f(x)}{h} \quad \text{exists for all } x \in \Omega.$$

$\Rightarrow$ f is analytic $\quad f(z) = \sum a_n z^n$

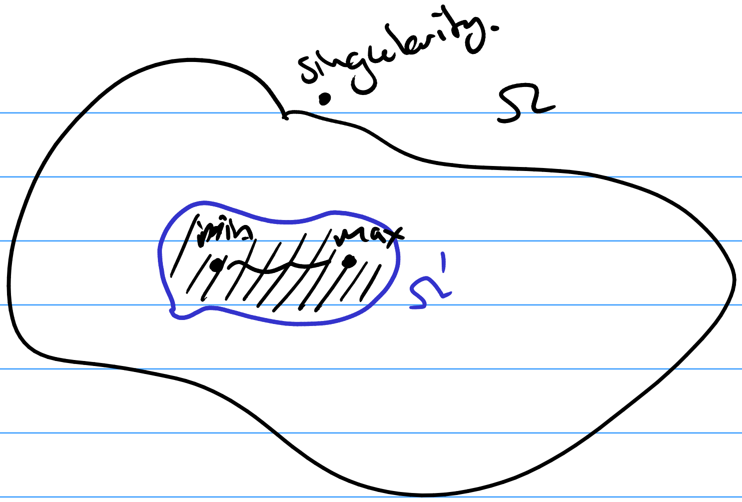
Cauchy-Riemann eqns $\quad f$ is holomorphic $\Leftrightarrow$ $f(x+iy) = u(x,y) + i v(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Notice

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

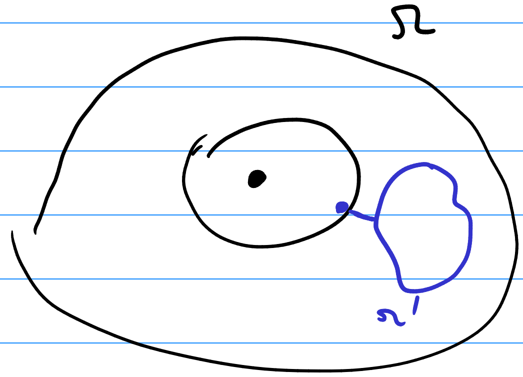
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



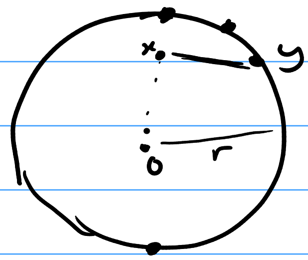
for any harmonic $u \geq 0$

$$\sup u \leq C(\Omega', \Omega) \inf u$$

$$\Phi = \frac{1}{|x|^{n-2}}$$



21. $\Omega = B(0,r) \qquad \Omega' = B(x, r')$



Poisson Representation formula = "super mean value formula"

$$u(x) = \frac{r^2 - |x|^2}{n r \omega_n} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} \, d\sigma(y)$$

If $y \in \partial B(0,r)$ $\qquad r - |x| \leq |x-y| \leq r + |x|$

$$(r-|x|)^{-n} \geq |x-y|^{-n} \geq (r+|x|)^{-n}$$

$$\int_{\partial B(0,r)} \frac{u(y)}{|x-y|} \, d\sigma(y) \leq \frac{1}{(r-|x|)^n} \int_{\partial B} u(y) \, d\sigma(y)$$

$$= \frac{1}{(r-|x|)^n} r^{n-1} \omega_n n \; u(0)$$

$$u(x) \leq \frac{\cancel{(r-|x|)}(r+|x|)}{\cancel{n r}\, \omega_n} \frac{1}{(r-|x|)^{n-1}} r^{n-2} \cancel{\omega_n n} \; u(0)$$

$$= r^{n-2} \frac{r+|x|}{(r-|x|)^{n-1}} u(0).$$

$$u(x) = \frac{1}{r^n \omega_n} \int_{B(x,r)} u(y)\,dy$$

Also the mean value property.

$$\int_0^R u(x)\,dr = \int_0^R \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)\,dy\,dr$$

$u(x)R$

$u \geqslant 0$

22.

a) $\dim P(d,n) = \binom{n+d-1}{d}$

$n=3$

$** \mid * \mid ***$ $\qquad$ $* \| **$

$x^2 \quad y \quad z^3$ $\qquad$ $x^1 y^0 z^2 = xz^2$

d stars $\quad n-1$ bars $\qquad \dfrac{(n+d-1)!}{d!\,(n-1)!} = \binom{n+d-1}{d}$

$p(\vec{x}) = x^2 y + y^2 z$ $\qquad$ homogeneous deg 3.

$p(\lambda x, \lambda y, \lambda z) = \lambda^3 p(x,y,z)$

b) $\Delta = \partial_1^2 + \partial_2^2 + \ldots + \partial_n^2$

$Q = x_1^{d_1} \cdot x_j^{d_j} \cdot x_n^{d_n}$

$\partial_j^2 = d_j(d_j-1)\, x_1^{d_1} \ldots x_j^{d_j-2} \ldots x_n^{d_n} = \begin{cases} 0 & \text{if } d_j = 0,1 \\ \deg & d-2 \end{cases}$

$\Delta Q =$ sum of 0's or deg $d-2$

$= 0$ or deg $d-2$.

$p(x,y) = 0$

$p(\lambda x, \lambda y) = 0 = \lambda^k 0 = \lambda^k p(x,y)$ $\quad$ for any k.

c) Show $\Delta : P(d+2,n) \to P(d,n)$ is surjective.

Induction on n:

- base case $n=1$: take $Q = x_1^d$ $\quad \Delta\left(\frac{1}{(d+1)(d+2)} x_1^{d+2}\right) = x_1^d$

If $p \in P(d,n)$ and $q = \Delta_{\mathbb{R}^n} p$.

$\Delta_{\mathbb{R}} = \frac{\partial^2}{\partial x_1^2} \qquad \Delta_{\mathbb{R}^n} = \partial_1^2 + \dots + \partial_n^2$

then also $q = \Delta_{\mathbb{R}^m} p$ for $m \geq n$.

$\Delta_{\mathbb{R}^2}(x_1^3) = \partial_1^2 (x_1)^3 + \cancel{\partial_2^2 (x_1^3)}$
$= \Delta_{\mathbb{R}^1}(x_1^3)$

ie for all $d \geq 0$ $\quad \Delta : P(d+2,1) \to P(d,1)$ is surj.

- Assume for all $n \leq N$ for all d, $\Delta : P(d+2,n) \to P(d,n)$ is surjective

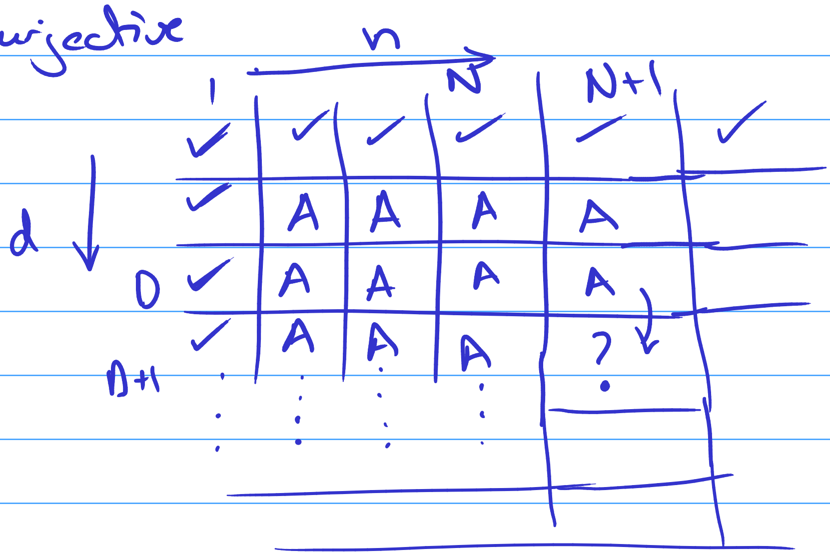
We want to show for all $d \geq 0$, $\Delta : P(d+2, N+1) \to P(d, N+1)$

Induction on d.

Base case is $d=0$. $\quad Q = 1$

$\Delta_{\mathbb{R}^{N+1}}(\frac{1}{2}x_1^2) = \partial_1^2 (\frac{1}{2}x_1^2) = 1 \quad ✓$

Assume for all $d \leq D$ that $\Delta : P(D+2, N+1) \to P(D, N+1)$ is surjective



Choose any monomial $Q \in P(D+1, N+1)$

$$Q = \underbrace{x_1^{d_1} \cdots\cdots x_N^{d_N}}_{P} \underbrace{x_{N+1}^{d_{N+1}}} = P\, x_{N+1}^{d_{N+1}}$$

$P \in P(D+1-d_{N+1}, N)$ $\quad \exists\, \tilde{P} \in P(D+3-d_{N+1}, N)$

$$\Delta_{\mathbb{R}^N} \tilde{P} = P = \Delta_{\mathbb{R}^{N+1}} \tilde{P} \qquad \partial_{N+1}^2 \tilde{P} = 0$$

$$\Delta_{\mathbb{R}^{N+1}}\left(\tilde{P} x_{N+1}^{d_{N+1}}\right) = \sum_{j=1}^{N} \partial_j \partial_j \left(\tilde{P} x_{N+1}^{d_{N+1}}\right) + \partial_{N+1}^2 \left(\tilde{P} x_{N+1}^{d_{N+1}}\right)$$

$$= \left(\Delta \tilde{P}\right) x_{N+1}^{d_{N+1}} + \tilde{P}\, d_{N+1}(d_{N+1}-1)\, x_{N+1}^{d_{N+1}-2}$$

$$= Q + \tilde{P}\, A\, x_{N+1}^{d_{N+1}-2}$$

$$Q = \Delta(\quad) + \Delta(\quad) \cdots\cdots + \underbrace{\tilde{\tilde{P}}_{\text{const}}\, x_{N+1}^{d_{N+1}-2-2-2-2\cdots}}_{0.}$$

$\Delta : P(d+2, N+1) \to P(d, N+1)$
is surjective for all d.

By induction holds for all n.

1. good examples.

2. Spherical harmonics. solving Poisson on S^n.

$$\Delta_{\mathbb{R}^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\Delta_{\mathbb{R}^n} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^n}\Delta_{S^n}$$

$x^2 - y^2$ on $\mathbb{R}^2$ $\quad r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$

in general $r^n Y(\theta_1, \theta_2, \dots \theta_n)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}r^n\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(n r^n\right) = n^2 r^{n-2}$$

$$0 = n^2 r^{n-2} Y + \Delta_{S^n} Y \qquad \text{on spher } r=1$$

$$\Delta_{S^n} Y = -n^2 Y$$

In $\mathbb{R}^2$ all $\quad r^k\cos(k\theta),\ r^k\sin(k\theta)$