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Introduction to Partial Differential Equations
Exercise sheet 3

14th October, 2020

6. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 point(s))

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.
.
(4 point(s))

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(2 point(s))*

Solution.

(a) By computation

$$\frac{d}{ds}z(s) = \nabla u(x(s)) \cdot \dot{x}(s) = \nabla u(x(s)) \cdot b(x(s)) = -c(x(s))u(x(s)) = -c(x(s))z(s).$$

(b) We follow the working at the beginning of Section 1.5 of the lecture script, specialising the argument to this particular case. As there, we have

$$\frac{dp}{ds} = \mathrm{Hess}(u)\dot{x} = \left(\sum_j \partial_i \partial_j u \, \dot{x}_j \right),$$

where $\mathrm{Hess}(u)$ is the matrix of second derivatives of u. The total derivative of F with respect to x is

$$\begin{aligned} 0 &= \partial_p F \cdot \partial_i p + \partial_z F \partial_i z + \partial_i F \\ &= b \cdot \partial_i p + \partial_z F \, p_i + \partial_i F \\ 0 &= \mathrm{Hess}(u) b + \partial_z F \, p + \nabla F. \end{aligned}$$

If we suppose that the characteristic has the property that $\dot{x} = b(z, x)$, then

$$\dot{p} = -\partial_z F\, p - \nabla F.$$

So $p(s)$ is described by an ODE and the assumption about $\dot{x}$ does not involve p. Finally,

$$\dot{z} = \nabla u \cdot \dot{x} = p \cdot \dot{x} = p \cdot b = -c(z, x)$$

since by assumption $F(p(s), z(s), x(s)) = 0$. This also does not depend on p. Hence we have the ODE for the characteristics, and the $\dot{x}$ and $\dot{z}$ equations do not depend on p.

(c) The inhomogeneous transport equation is defined by

$$F(p, z, x) = \tilde{b} \cdot p - f(\tilde{x})$$

where $\tilde{x} = (x, t)$ and $\tilde{b} = (b, 1)$ in $\mathbb{R}^{n+1}$. From the equations we have just derived, we see that $\dot{\tilde{x}}(s) = (b, 1)$ tells us that the characteristic lines are straight lines $\tilde{x}(s) = (bs + x_0, s)$. Or in non-parametric form $x = bt + x_0$. The next ODE is $\dot{z}(s) = f(x(s), s) = f(x_0 + bs, s)$. This too can be directly integrated now

$$z(t) - z(0) = \int_0^t \dot{z}(s)\, ds = \int_0^t f(x_0 + bs, s)\, ds = \int_0^t f(x - bt + bs, s)\, ds.$$

Together with the initial condition $z(0) = u(x(0), 0) = u(x_0, 0) = g(x - bt)$ this is exactly the solution that we found previously.

7. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_1 \partial_1 u + x_2 \partial_2 u = 2u$ on the domain $x_1 \in \mathbb{R}, x_2 > 0$, with initial condition $u(x_1, 1) = g(x_1)$.

(4 point(s))

(b) $x_1 \partial_2 u - x_2 \partial_1 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$.

(4 point(s))

(c) $x_1 \partial_1 u + 2x_2 \partial_2 u + \partial_3 u = 3u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$.

(4 point(s))

(d) $u \partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2} x_1$.

(5 point(s))

Solution. These PDEs are all of the linear type of the previous question, so we can use the ODEs for the characteristics that we have already derived.

(a) Write the PDE as $(x_1, x_2) \cdot p - 2z = 0$. The the ODE for $\dot{x}$ is $\dot{x}_i(s) = x_i$, which is solved by the exponential $x(s) = e^s(a, 1)$. These are the characteristics. Likewise $\dot{z} = 2z$ gives $z(s) = e^{2s}z(0) = e^{2s}u(a, 1)$, which tells us the value of u on those lines. Now we must apply the initial condition. Given a point (x_1, x_2) it must lie on the same characteristic as $(x_1/x_2, 1)$, which is easy to see if you rescale the parameter $r = e^s > 0$. Then

$$u(e^s(a, 1)) = u(x(s)) = z(s) = e^{2s}u(a, 1)$$
$$u(x_1, x_2) = u(x_2(x_1/x_2, 1)) = (x_2)^2 u(x_1/x_2, 1) = x_2^2 g(x_1/x_2).$$

(b) This PDE is $(-x_2, x_1) \cdot p - z = 0$. The system of ODEs therefore reads in part

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1,$$

which is the well know system solved by the sinusoidal functions. From the boundary $(x_1, 0)$ we see that $x_2 = 0$ when $s = 0$. Therefore $x_2 = r \sin s$ and $x_1 = r \cos s$ for a constant r. The ODE describing the values of u is $\dot{z} = z$, so $u(x(s)) = e^s u(x(0)) = e^s u(r, 0) = e^s g(r)$. Solving for the parameters r, s in terms of the point x gives

$$u(x) = e^{\arctan(x_2/x_1)} g(|x|).$$

(c) This is quite similar to part (a), albeit with more variables. From $F = (x_1, 2x_2, 1) \cdot p - 3z$ it follows that

$$x(s) = (x_{10}e^s, x_{20}e^{2s}, x_{30} + s) = (ae^s, be^{2s}, s).$$

Already we can determine the appropriate parameter values for any point: $s = x_3$, $a = x_1 e^{-x_3}$, and $b = x_2 e^{-2x_3}$. The the equation for the values is $z = z(0)e^{3s}$, so

$$u(x) = e^{3s}u(a, b, 0) = e^{3x_3} g\left(x_1 e^{-x_3}, x_2 e^{-2x_3}\right).$$

(d) This PDE, $F = (z, 1) \cdot p - 1$ is a little different to the others, because of the z in the coefficients of p. This creates a linkage in the system of ODEs:

$$\dot{x}_1 = z, \quad \dot{x}_2 = 1, \quad \dot{z} = 1.$$

Fortunately, we can solve for z first this time quite easily: $z(s) = s + z(0)$. Then $x(s) = (\frac{1}{2}s^2 + sz(0) + a, s + a)$, using the fact that the initial boundary is (a, a). Hence we can say that $z(0) = u(x(0)) = u(a, a) = \frac{1}{2}a$ and $a = x_2 - s$, which allows us to solve for s:

$$\begin{aligned} x_1 &= \frac{1}{2}s^2 + s\frac{1}{2}(x_2 - s) + x_2 - s \\ x_1 - x_2 &= \frac{1}{2}x_2 s - s \\ s &= \frac{2x_1 - 2x_2}{x_2 - 2}. \end{aligned}$$

Finally, what we are interested in is the value of the solution u on these curves, and $u(x(s)) = z(s) = s + z(0) = s + \frac{1}{2}(x_2 - s)$, ie

$$u(x) = \frac{1}{2}x_2 + \frac{1}{2}\frac{2x_1 - 2x_2}{x_2 - 2}.$$

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.