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Introduction to Partial Differential Equations
Exercise sheet 8

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24. Do nothing by halves.

Let $H^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ be the upper half-space and $H^0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ the dividing hyperplane. We call $R(x) = (x_1, \ldots, x_{n-1}, -x_n)$ reflection in the plane H^0. Similarly $B^+ = B(0,1) \cap H^+$ and $B^0 = B(0,1) \cap H^0$.

(a) A reflection principle for harmonic functions. Let $u : \overline{B^+} \to \mathbb{R}^n$ be a harmonic function with $u|_{B^0} = 0$. Show that the function $v : \overline{B} \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_n \geq 0 \\ -u(R(x)) & \text{for } x_n < 0 \end{cases}$$

is harmonic. *(4 Point(s))*

(b) Green's function for the upper half-space. Show that Green's function for H^+ is

$$G(x,y) = \Phi(x-y) - \Phi(R(x)-y).$$

(3 Point(s))

(c) Green's function for the half-ball. Compute the Green's function for B^+.
Hint: Make use of both the Green's function for the ball 3.20 and part (b).
(3 Point(s))

Solution.

(a) One could try to show directly that v is harmonic. Clearly it is when $x_n \neq 0$, and it is possible to compute the necessary derivatives when $x_n = 0$. However, there is a more general method using the uniqueness of the solution to the Dirichlet problem on the ball.

Let $g = v|_{\partial B}$ be the restriction to this function on the sphere. This is continuous, in particular when $x_n = 0$. There is a unique solution $\tilde{v}$ to the Laplace equation $\Delta \tilde{v} = 0$ with $\tilde{v}|_{\partial B} = g$. We see that $-\tilde{v} \circ R$ is also a solution to this equation, thus $\tilde{v} = -\tilde{v} \circ R$. This implies that $\tilde{v}$ vanishes on B^0.

Now consider $\tilde{v} - u$. This is also a harmonic function, and moreover it is identically zero on ∂B^+. The maximum principle says it has to be zero on all of B^+. Thus $\tilde{v} = u$ on B^+ and by reflection $\tilde{v} = v$ on $\overline{B}$.

(b) Let Φ be the fundamental solution to the Laplace equation. Let $G(x,y)$ be the Green's function for H^+. The required properties are (1) that for any $x \in H^+$ the function $G(x,y) - \Phi(x-y)$ is a harmonic function of y and (2) that for any $x \in H^+$ we have $\lim_{y \to H^0} G(x,y) = 0$. We saw for the unit ball that the Greens function was a difference of the fundamental solution and its reflection across the boundary of the ball. That way, the two cancelled on the boundary and gave the second property. So let use try

$$G(x,y) = \Phi(x-y) - \Phi(R(x)-y).$$

The first property is satisfied because $\Phi(R(x)-y)$ is only not harmonic when $y=R(x)$, and for $x \in H^+$ this only occurs when $y \in H^-$. To show the second property, note that $\Phi(z)$ is radially symmetric. Since $\|R(x)-y\| = \|x-R(y)\|$, for any $y \in H^0$ we have

$$G(x,y) = \Phi(x-y) - \Phi(x-R(y)) = \Phi(x-y) - \Phi(x-y) = 0.$$

Thus we have shown that G is the Greens function.

(c) To discuss the Greens function for the half-ball, we should introduce a symbol for inversion in the sphere, $\iota(x) := |x|^{-2}x$. We know from lectures then

$$G_B(x,y) = \Phi(x-y) - \Phi(|x|(\iota(x)-y)).$$

Following the ideas of the previous question, we guess that the Greens function for the half-ball is the reflection of this one

$$G(x,y) = \Phi(x-y) - \Phi(|x|(\iota(x)-y)) - \Phi(R(x)-y) + \Phi(|x|(\iota(R(x))-y)).$$

If both $x, y \in B^+$ then $\iota(x)-y$, $R(x)-y$, and $|x|(\iota(R(x))-y)$ are never zero, so $G(x,y) - \Phi(x,y)$ is harmonic. The boundary of B^+ has two parts $\overline{B^0}$ and $\partial B^+ \cap H^+$. If $y \in \overline{B^0}$ then

$$\begin{aligned} G(x,y) &= \Phi(x-y) - \Phi(|x|(\iota(x)-y)) - \Phi(x-R(y)) + \Phi(|x|(R(\iota(x))-y)) \\ &= \Phi(x-y) - \Phi(|x|(\iota(x)-y)) - \Phi(x-y) + \Phi(|x|(\iota(x)-y)) = 0. \end{aligned}$$

On the other hand, if $y \in \partial B^+ \cap H^+$ is in the hemispherical part, then $\||x|(\iota(x)-y)\| = \|x-y\|$ as in the lecture notes, but also $\||x|(\iota(R(x))-y)\| = \|R(x)-y\|$, so

$$G(x,y) = [\Phi(x-y) - \Phi(|x|(\iota(x)-y))] - [\Phi(R(x)-y) - \Phi(|x|(\iota(R(x))-y))] = 0.$$

25. Teach a man to fish...

Using the Green's function of H^+ from the previous question, derive a formal integral representation for a solution of the Dirichlet problem

$$\Delta u = 0 \text{ in } H^+, \qquad u|_{H^0} = g.$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined.

(5 Point(s))

Solution. Begin with Greens Representation formula

$$u(x) = -\int_{H^+} G_{H^+}(x,y)\Delta_y u(y)\,\mathrm{d}y - \int_{H^0} u(z)\nabla_z G_{H^+}(x,z)\cdot N\,\mathrm{d}\sigma(z).$$

The function u is harmonic, so the first integral vanishes. For the second term, $\nabla_z G_{H^+}(x,z) = -\nabla_z\Phi(x-z) + \nabla_z\Phi(R(x)-z)$ and we already computed the gradient of the fundamental solution in Theorem 3.2: $\nabla\Phi(y) = -\frac{1}{n\omega_n}\frac{y}{|y|^n}$. The normal is also easy to describe, it points in the negative x_{n+1} direction: $N = (0, \ldots, 0, -1)$. Therefore

$$\begin{aligned} u(x) &= -\int_{H^0} u(z)\frac{1}{n\omega_n}\left[\frac{x-z}{|x-z|^n} - \frac{R(x)-z}{|R(x)-z|^n}\right] \cdot (0, \ldots, 0, -1)\,\mathrm{d}\sigma(z) \\ &= \frac{1}{n\omega_n}\int_{H^0} u(z)\left[\frac{x_n - z_n}{|x-z|^n} - \frac{-x_n - z_n}{|x-z|^n}\right]\,\mathrm{d}\sigma(z) \\ &= \frac{2x_n}{n\omega_n}\int_{H^0}\frac{u(z)}{|x-z|^n}\,\mathrm{d}\sigma(z). \end{aligned}$$

26. An alternative estimate for Corollary 3.4.

(a) Show the following estimate for all $x \neq 0$ and multiindices α:

$$|\partial^\alpha\,|x|^{-n}| \le A(n, |\alpha|)\,|x|^{-n-|\alpha|},$$

where $A(n, |\alpha|)$ is a constant depending only on n and order $|\alpha|$. *(4 Point(s))*

(b) Hence give an alternative proof of Corollary 3.4 (you do not have to give a particular form for the constant).
Hint: Start from Poisson's representational formula. *(4 Point(s))*

Solution.

(a) We know (in a heuristic sense) that harmonic functions are limited in how fast they can grow. So we develop a bounds for the reciprocal of powers of the absolute value. Observe the first derivative:

$$\partial_k|x|^{-n} = -\frac{n}{2}\cdot 2x_k(x_1^2 + \cdots + x_n^2)^{-n/2-1} = -nx_k\,|x|^{-n-2}.$$

We hypothesise then that the α-derivative has the form $P_\alpha\,|x|^{-n-2|\alpha|}$. Indeed

$$\begin{aligned} \partial_k(P_\alpha|x|^{-n-2|\alpha|}) &= P_\alpha\cdot -(n+2|\alpha|)x_k\,|x|^{-n-2|\alpha|-2} + \partial_k P_\alpha|x|^{-n-2|\alpha|} \\ &= \left(-(n+2|\alpha|)P_\alpha x_k + \partial_k P_\alpha|x|^2\right)|x|^{-n-2|\alpha+e_k|}. \end{aligned}$$

So we see that the form is preserved with the recurrence relation $P_{\alpha+e_k} = -(n+2|\alpha|)P_\alpha x_k + \partial_k P_\alpha|x|^2$. Together with $P_{e_k} = -nx_k$ this shows that the P_α are all polynomials. Moreover, these polynomials are homogeneous, since both terms of the recurrence relation increase the degree by 1. So the homogeneous degree of P_α is $|\alpha|$.
It is reasonable, and necessary for the question, to bound this by $|P_\alpha(x)| \le \tilde{A}(n,\alpha)|x|^{|\alpha|}$. Since $|x_k| \le |x|$, this follows immediately from the triangle inequality and homogeneity, with $\tilde{A}(n,\alpha)$ being the sum of the absolute values of the coefficients of P_α. Putting this all together

$$|\partial^\alpha|x|^{-n}| = |P_\alpha|\,|x|^{-n-2|\alpha|} \le \tilde{A}(n,\alpha)|x|^{|\alpha|}|x|^{-n-2|\alpha|} = \tilde{A}(n,\alpha)|x|^{-n-|\alpha|}.$$

Letting $A(n,k)$ be the maximum of $\tilde{A}(n,\alpha)$ for $|\alpha| = k$ gives a constant that only depends the order of the derivative.

(b) For harmonic functions, Poisson's representational formula reads

$$u(x) = \frac{r^2 - |x-a|^2}{nr\omega_n} \int_{\partial B(a,r)} \frac{u(y)}{|x-y|^n} \, \mathrm{d}\sigma(y)$$

where we have assumed, without loss of generality, that the ball is centred at 0. Applying ∂^α to this requires Leibniz's rule, but the first factor has at most two derivatives. Therefore we have the following unusual form

$$\begin{aligned}
|\partial^\alpha u(x)| &\le \frac{|r^2 - |x-a|^2|}{nr\omega_n} \int_{\partial B(a,r)} \left| \partial^\alpha \frac{u(y)}{|x-y|^n} \right| \mathrm{d}\sigma(y) \\
&\quad + \sum_{k \,:\, \alpha_k \ge 1} \frac{|-2(x_k - a_k)|}{nr\omega_n} \int_{\partial B(a,r)} \left| \partial^{\alpha - e_k} \frac{u(y)}{|x-y|^n} \right| \mathrm{d}\sigma(y) \\
&\quad + \sum_{k \,:\, \alpha_k \ge 2} \frac{|-2|}{nr\omega_n} \int_{\partial B(a,r)} \left| \partial^{\alpha - 2e_k} \frac{u(y)}{|x-y|^n} \right| \mathrm{d}\sigma(y) \\
&\le \Bigg[\frac{|r^2 - |x-a|^2|}{nr\omega_n} A(n, |\alpha|) \int_{\partial B(a,r)} |x-y|^{-n-|\alpha|} \, \mathrm{d}\sigma(y) \\
&\quad + \sum_k \frac{2|x-a|}{nr\omega_n} A(n, |\alpha| - 1) \int_{\partial B(a,r)} |x-y|^{-n-|\alpha|+1} \, \mathrm{d}\sigma(y) \\
&\quad + \sum_k \frac{2}{nr\omega_n} A(n, |\alpha| - 2) \int_{\partial B(a,r)} |x-y|^{-n-|\alpha|+2} \, \mathrm{d}\sigma(y) \Bigg] \|u\|_{L^\infty}.
\end{aligned}$$

This bound holds for any value of a, so choose it to be the point where we are trying to estimate the derivative, namely $a = x$. The main consequences are that $|x - y|$ becomes the constant value r for $y \in \partial B(x, r)$ and the middle set of terms vanishes. Hence

$$\begin{aligned}
|\partial^\alpha u(x)| &\le \Bigg[\frac{r}{n\omega_n} A(n, |\alpha|) \int_{\partial B(x,r)} r^{-n-|\alpha|} \, \mathrm{d}\sigma(y) \\
&\quad + \sum_k \frac{2}{nr\omega_n} A(n, |\alpha| - 2) \int_{\partial B(a,r)} r^{-n-|\alpha|+2} \, \mathrm{d}\sigma(y) \Bigg] \|u\|_{L^\infty} \\
&= \left[\frac{1}{n\omega_n} A(n, |\alpha|) r^{-n-|\alpha|+1} n\omega_n r^{n-1} + \sum_k \frac{2}{n\omega_n} A(n, |\alpha| - 2) r^{-n-|\alpha|+1} n\omega_n r^{n-1} \right] \|u\|_{L^\infty} \\
&= \left[A(n, |\alpha|) + \sum_k 2A(n, |\alpha| - 2) \right] r^{-|\alpha|} \|u\|_{L^\infty}.
\end{aligned}$$

This proves the bound.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.