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Introduction to Partial Differential Equations
Exercise sheet 4

21st October, 2020

8. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C x \, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 point(s))*

(b) Consider the parametrisation of the circle $t \mapsto (\cos t, \sin t)$. Compute the integral in this parametrisation. *(2 point(s))*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1 - x^2})$. Compute the integral in this parametrisation. *(2 point(s))*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. [Hint. The easiest way is to use two parametrisations similar to part (b)] *(2 point(s))*

(e) Compute this integral using the divergence theorem. *(3 point(s))*

Solution.

(a) This depends somewhat on the definition of integral that you are using. In Lebesgue integration, sets of measure zero can no contribute to the final value, and a finite collection of points is measure zero in dimensions 1 and higher. With Darboux or Riemann integrations, these are defined initially on closed sets only. They are extended to open sets, or in this case punctured neighbourhoods by taking a limit of closed sets. Continuity of the integrand is sufficient then to ensure there is no difference.

(b) Using the previous part, we know that we can integrate from $t = 0$ to $t = 2\pi$, since counting the starting/ending point twice does not affect the value of the integral.

We must also calculate the area-element factor. The coordinate maps $\Phi : (0, 2\pi) \subset \mathbb{R} \to \mathbb{R}^2$, so its derivative is size 2×1, namely $(-\sin t, \cos t)^T$. The factor therefore is

$$\det \begin{bmatrix} -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \det \begin{bmatrix} 1 \end{bmatrix} = 1$$

We can now compute the integral

$$\int_0^{2\pi} \cos t \times 1 \, dt = \sin t \Big|_0^{2\pi} = 0.$$

(c) Here we have that $\Phi_1' = (1, -x(1 - x^2)^{-0.5})^T$. So

$$\int_{U_1} x \, d\sigma = \int_{-1}^{1} x\sqrt{1 + x^2(1 - x^2)^{-1}} \, dx = \int_{-1}^{1} x(1 - x^2)^{-0.5} \, dx = -(1 - x^2)^{0.5}\Big|_{-1}^{1} = 0.$$

And likewise for U_2.

(d) As hinted at, start with the parametrisation Φ defined in part (b). Now take a bump function h on $(0, 2\pi)$ such that it is identically 1 on $[\pi/2, 3pi/2]$ and has compact support K strictly contained in the interval. Now, we need a second coordinate chart to cover the point $t = 0$, ie $(1, 0) \in C$. For this we use $\Psi : (-\pi/2, \pi/2)$ given by $\Psi(t) = (\cos t, \sin t)$. Because it has the same formula, the area-element of Ψ is also 1.

Notice that $V_1 = C \setminus \{(1, 0)\}$, $h_1 = h$, $V_2 = C \cap \{x > 0\}$, and $h_2 = 1 - h$ is a partition of unity for C, and that Φ and Ψ are coordinates for the two set respectively. Hence

$$\begin{aligned}
\int_C x \, d\sigma &= \int_0^{2\pi} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t)) \cos t \, dt \\
&= \int_{\pi/2}^{3\pi/2} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t)) \cos t \, dt \\
&= \int_{\pi/2}^{3\pi/2} \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t) + h(t)) \cos t \, dt \\
&= \int_{-\pi/2}^{3\pi/2} \cos t \, dt,
\end{aligned}$$

which comes out to the same calculation as in part (b). Thus we see that a partition of unity is really a nice technical tool to divide an integral into pieces without worrying about overlaps and/or missing points.

(e) To apply the divergence theorem, we recognise C as the boundary of the disc $\Omega = \{x^2 + y^2 \leq 1\}$, with the outward pointing normal $N = (x, y)$. (Because this it is the unit circle, this normal N is already unit length.) We now need to write the integrand x in the form $f \cdot N = (f_1, f_2) \cdot N = x f_1 + y f_2$. We see that $f = (1, 0)$ fits. Therefore

$$\int_C x \, d\sigma = \int_\Omega \nabla \cdot f \, d\mu = \int_\Omega 0 \, d\mu = 0.$$

9. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the first Green formula

$$\int_\Omega v \triangle u \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \nabla u \cdot N \, d\sigma.$$

(3 points)

(b) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v \triangle u - u \triangle v) \, dx = \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \, d\sigma.$$

(1 points)

Solution.

(a) The final term of the formula can have the divergence theorem applied to it:

$$-\int_{\Omega} \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} v \nabla u \cdot N \; d\sigma = \int_{\Omega} -\nabla u \cdot \nabla v + \nabla \cdot (v \nabla u) \; dx$$

If g is a scalar-valued function and F is a vector-valued function, recall the product rule (or derive it for yourself): $\nabla \cdot (gF) = (\nabla g) \cdot F + g \nabla \cdot F$. Applying this formula here, and remembering that the Laplacian is the divergence of the gradient, gives

$$\int_{\Omega} -\nabla u \cdot \nabla v + \nabla \cdot (v \nabla u) \; dx = \int_{\Omega} -\nabla u \cdot \nabla v + \nabla v \cdot \nabla u + v \Delta u \; dx = \int_{\Omega} v \Delta u \; dx.$$

(b) The second Greens formula is simply a symmetrised version of the first:

$$\begin{aligned}\int_{\Omega} (v \triangle u - u \triangle v) \; dx &= -\int_{\Omega} (\nabla u \cdot \nabla v - \nabla v \cdot \nabla u) \; dx + \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \; d\sigma \\ &= \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \; d\sigma.\end{aligned}$$

10. The Black Spot.

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

(a) Show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N \; d\sigma$$

does not depend on the radius r, where N is the outward pointing normal. *(3 points)*

(b) What property of g explains this fact? In your proof, be careful to note that g is singular at $(0, 0)$. *(3 points)*

(c) Prove for any compact region $\Omega \subset \mathbb{R}^2$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \; d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

(d) Comment on the flux of ∇g. *(1 points)*

Solution.

(a) The outward pointing unit normal of the ball of radius r is $N = r^{-1}(x, y)$, where $r = \sqrt{x^2 + y^2}$ is constant on ∂B_r. The gradient $\nabla g = r^{-2}(2x, 2y)$. Together we then have

$$\int_{\partial B_r} \nabla g \cdot N \; d\sigma = \int_{\partial B_r} r^{-3}(2x^2 + 2y^2) \; d\sigma = 2r^{-1} \int_{\partial B_r} 1 \; d\sigma.$$

Now, the integral above is just the 1 times the circumference of the circle, and therefore the value is $2r^{-1} \times 2\pi r = 4\pi$.

(b) Let us consider the difference of two of these integrals for different radii $r < R$.

$$\int_{\partial B_R} \nabla g \cdot N \, d\sigma - \int_{\partial B_r} \nabla g \cdot N \, d\sigma = \int_{\partial B_R} \nabla g \cdot N \, d\sigma + \int_{\partial B_r} \nabla g \cdot (-N) \, d\sigma = \int_{\partial A_{r,R}} \nabla g \cdot N \, d\sigma,$$

where $A_{r,R}$ is the annulus with inner radius r and outer radius R. Note, the boundary of the annulus consists of two disjoint circles and the outward pointing normal of the annulus on the inner boundary circle is the outward point normal of the disc B_r. This explains the sign in the above calculation.

If we apply the divergence theorem to the annulus, we get

$$\int_{\partial A_{r,R}} \nabla g \cdot N \, d\sigma = \int_{A_{r,R}} \Delta g \, dx,$$

and the Laplacian of g is zero:

$$\begin{aligned}\Delta g &= \frac{\partial}{\partial x} \frac{2x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{2y}{x^2 + y^2} \\ &= \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \\ &= 0.\end{aligned}$$

Therefore the integral over the annulus is zero, and hence the difference of the integrals on the two circles is also zero.

If we are being precise, the Laplacian of g is zero on $\mathbb{R}^2 \setminus \{(0,0)\}$; at the origin it is not defined/singular. This is why we could not apply the divergence theorem directly to B_r.

(c) Suppose first that $(0,0) \notin \Omega$. We have already seen that the Laplacian of g is zero, and we can immediately conclude that the integral over $\partial\Omega$ is zero too.

If however $(0,0)$ lies in the interior of Ω, we must first excise it. There exists some small ϵ such that $B_\epsilon \subset \Omega$. Then

$$\begin{aligned}\int_{\partial\Omega} \nabla g \cdot N \, d\sigma &= \int_{\partial B_\epsilon} \nabla g \cdot N \, d\sigma + \left(\int_{\partial\Omega} \nabla g \cdot N \, d\sigma - \int_{\partial B_\epsilon} \nabla g \cdot N \, d\sigma \right) \\ &= 4\pi + \int_{\Omega \setminus B_\epsilon} \Delta g \, dx \\ &= 4\pi.\end{aligned}$$

(d) If we consider a region that does not contain the origin, then the flux of ∇g through this region is zero; there is as much entering as leaving. However, if the region does contain the origin, then there is an outward flow of 4π. We can interpret this as saying that there is a source of material at the origin, and that material is otherwise conserved as it flows. We see from the particular form of g that this flow is outward and radially symmetric.

11. Convoluted.

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\, dy.$$

(a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. *(2 Points)*

(b) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(2+3+4 Points)*

Solution.

(a) Immediately,

$$f * f(x) = \int_{\mathbb{R}} f(y)f(x-y)\, dy = \int_{-1}^{1} f(x-y)\, dy.$$

Now, $f(x-y)$ is only non-zero when $x - y \in [-1, 1]$, ie when $x - 1 \leq y \leq x + 1$. Therefore

$$f * f(x) = \begin{cases} 0 & \text{if } x < -2 \\ x+2 & \text{if } -2 \leq x < 0 \\ 2-x & \text{if } 0 \leq x < 2 \\ 0 & \text{if } 2 \leq x. \end{cases}$$

(b) Formal bilinearity follows from the linearity of the integral and the bilinearity of the product of functions. The smoothness and compact support of all functions involved means that the integrals always exist.

Commutativity:

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy = \int_{\infty}^{-\infty} f(x-z)g(z)\, (-dz) = g * f(x).$$

Associativity

$$\begin{aligned} f * (g * h)(x) &= \int_{-\infty}^{\infty} f(y)(g * h)(x-y)\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(z)h((x-y)-z)\, dz\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(z)h(x-(y+z))\, dz\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(w-y)h(x-w)\, dy\, dw \\ &= \int_{-\infty}^{\infty} (f * g)(w)h(x-w)\, dw \\ &= (f * g) * h(x) \end{aligned}$$

12. Go with the flow.

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity ρv, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold S is the integral

$$\int_S \rho v \cdot N \, d\sigma,$$

where N is the normal of S.

(a) Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

(b) How does this relate to the form of the conservation law derived in the lectures?

(c) For liquids a common property is *incompressibility*. For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that ρ is constant. However, slightly more general model says that ρ is not globally constant, but if we follow a point $x(t)$ along the velocity field v then $\rho(x(t),t)$ is constant. An example would be oil and water mixed together.
Use this description of incompressible flow to show that $\nabla \cdot v = 0$.

Solution.

(a) By defining flux in the way we have, the divergence theorem applies. Let S be a surface enclosing a volume V:

$$\int_S \rho v \cdot N \, d\sigma = \int_V \nabla \cdot (\rho v) \, dx.$$

On the other hand, the amount of substance in V is the integral of ρ. Conservation means that the (positive) change of substance should be equal to the negative of the outward flux:

$$\frac{\partial}{\partial t} \int_V \rho \, dx = -\int_S \rho v \cdot N \, d\sigma \Rightarrow \int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) \, dx = 0.$$

Since this should hold for every volume V, we conclude that the integrand must be identically zero. We will give a rigorous proof of this final step in a few weeks time.

(b) The conservation law in lectures was only for one-dimensional situations, so $\nabla \cdot$ is the same as ∂_x. In the previous part we could also have an arbitrary flux function f instead of only the form ρv and still apply the same working to arrive at

$$\frac{\partial \rho}{\partial t} + \nabla \cdot f = 0 \Rightarrow \dot{\rho} + \partial_x f = 0.$$

If the flux function only depends on the density ρ, and not on the coordinate, then we arrive exactly at the form in Theorem 1.10.

(c) The condition of being constant along a flow is very similar to the idea behind the method of characteristics. If we take the total derivative

$$0 = \frac{d}{dt}\rho = \nabla\rho \cdot \dot{x} + \dot{\rho} = \nabla\rho \cdot v + \dot{\rho}.$$

On the other hand, we can expand the conservation law

$$0 = \dot{\rho} + \nabla \cdot (\rho v) = \dot{\rho} + \nabla\rho \cdot v + \rho\nabla \cdot v = \rho\nabla \cdot v.$$

So either there is no substance (which is trivial) or $\nabla \cdot v = 0$ as required.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.