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Introduction to Partial Differential Equations
Exercise sheet 6

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17. Preparing the Mean Value Theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$, and $\partial B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$ for $r > 0$. Show that the function

$$F(r) := \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) \, \mathrm{d}\sigma(x)$$

converges to $f(x_0)$ as $r \to 0$. *(4 Point(s))*

Solution. $F(r)$ is a single variable function and so the limit is just in the ordinary sense:

$$\begin{aligned}
|F(r) - f(x_0)| &= \left| \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) \, \mathrm{d}\sigma(x) - f(x_0) \right| \\
&= \left| \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) \, \mathrm{d}\sigma(x) - f(x_0) \times \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} \mathrm{d}\sigma(x) \right| \\
&= \left| \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) - f(x_0) \, \mathrm{d}\sigma(x) \right| \\
&\leq \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} |f(x) - f(x_0)| \, \mathrm{d}\sigma(x) \\
&\leq \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} \sup_{x \in \partial B(0,r)} |f(x) - f(x_0)| \, \mathrm{d}\sigma(x) \\
&= \sup_{x \in \partial B(0,r)} |f(x) - f(x_0)| \, .
\end{aligned}$$

This upper bound is also a function of r and by the continuity of f it converges to 0 as $r \to 0$. Therefore $F(r)$ converges to $f(x_0)$ as required.

18. Harmonic Functions on $B(0, 1) \subset \mathbb{R}^2$.

Let B be the open unit disc in $\mathbb{R}^2$.

(a) Let $u \in C^2(\overline{B})$ be a harmonic function which is given in polar coordinates $u = u(r, \varphi)$, with $0 \leq r \leq 1$ and $-\pi < \varphi \leq \pi$. Show that in this coordinates

$$\int_{\partial B} \frac{\partial u}{\partial r} \, \mathrm{d}\sigma = 0.$$

(2 Point(s))

(b) "Guess" a solution $u \in C^2(\overline{B})$ for each of the following *Neumann Problems* or show that there is no solution

(i) $\Delta u = 0$ on B with $\frac{\partial u}{\partial r} = \sin \varphi$ on ∂B.

(ii) $\Delta u = 0$ on B with $\frac{\partial u}{\partial r} = \sin^2 \varphi$ on ∂B.

(4 Point(s))

Solution.

(a) First, let's compute the Laplacian in polar coordinates, for later use. The change of variables is $x = r\cos\varphi, y = r\sin\varphi$ and the chain rule states

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} = \cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}\\ \frac{\partial}{\partial y} &= \sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}\\ \Delta &= \frac{\partial^2}{(\partial r)^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{(\partial\varphi)^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{(\partial\varphi)^2}.\end{aligned}$$

Note that in polar coordinate the outward pointing normal of the ball is simply $(1,0)$ because it is radial and length 1. Therefore we see that the integral is already in the form of the divergence theorem:

$$\int_{\partial B}\frac{\partial u}{\partial r}\,\mathrm{d}\sigma = \int_{\partial B}\nabla u\cdot N\,\mathrm{d}\sigma = \int_B \Delta u\,\mathrm{d}\sigma = 0.$$

(b) **(i)** For this question we guess that the solution is of the form $u(r,\varphi) = f(r)\sin\varphi$. Laplace's equation then reads

$$\left(f'' + \frac{1}{r}f'\right)\sin\varphi - \frac{1}{r^2}f\sin\varphi = 0 \Rightarrow r^2 f'' + rf' - f = 0 \Rightarrow f(r) = Ar^{-1} + Br.$$

Since we want the solution to be defined on the disc, we must choose $A = 0$. The boundary condition $\partial_r u|_{\partial B} = \sin\varphi$ then requires that $f'(1) = B = 1$. Hence the solution is $u(r,\varphi) = r\sin\varphi$.

Writing this in Cartesian coordinates make it trivial to see why this is harmonic: $u(x,y) = y$.

(ii) There can be no such function in this case because it would not obey the property established in part (a).

19. Inside Out.

(Optional) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be defined by

$$g(x) := \frac{1}{|x|^{n-2}} f\left(\frac{x}{|x|^2}\right).$$

Express Δg in terms of Δf, and thereby conclude that if f is a harmonic function then so too is g.

Solution. It helps to first have the x_k-partial derivative of $|x|$:

$$\partial_k |x|^2 = \partial_k(x_1^2 + \cdots + x_n^2) = 2x_k \Rightarrow \partial_k|x| = |x|^{-1}x_k.$$

And then I guess you have to differentiate twice and everything cancels nicely. But I haven't done this.

20. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A twice continuously differentiable function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if $-\Delta v \leq 0$ in Ω.

(a) Let $v : \overline{\Omega} \to \mathbb{R}$ be subharmonic. Show that for all $x \in \Omega$ and $r > 0$ with $B(x, r) \subset \Omega$:

$$v(x) \leq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v(y) \, \mathrm{d}\sigma(y).$$

Hint: Show the function from Question 17 is monotonic. *(3 Point(s))*

(b) Conclude therefore the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 Point(s))*

(c) Now let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function.

(i) Show that $\|\nabla u\|^2$ is subharmonic. *(2 Point(s))*

(ii) Show that $f \circ u$ is subharmonic for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$. *(2 Point(s))*

Solution.

(a) Fix a point $x_0 \in \Omega$. Let ω_n be the n-volume of the unit n-ball. It is well known, and follows by a simple calculus argument that the surface $(n-1)$-measure of the unit n-ball is $n\omega_n$. Following the hint, we first consider the averaging function

$$V(r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(x) \, \mathrm{d}\sigma(x) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} v(x_0 + ry) \, \mathrm{d}\sigma(y).$$

Its derivative is therefore

$$V'(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \partial_r v(x_0 + ry) \, \mathrm{d}\sigma(y) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta v(x_0 + ry) \, \mathrm{d}y,$$

similar to the question *Harmonic Functions on* $B(0, 1)$. Because the Laplacian of v is nonnegative, so too is this integral. This show that $V(r)$ is an (non-strictly) increasing function. On the other hand, we know that $v(x_0)$ is the limit of $V(r)$ as $r \to 0$. It must therefore be that $v(x_0) \leq V(r)$ for $r > 0$ and $B(x_0, r) \subset \Omega$.

(b) Suppose that v does indeed obtain a maximum at some $x_0 \in \Omega$ and that $B(x_0, R) \subset \Omega$. By the previous part we have that

$$0 \geq v(x_0) - \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(x) \, \mathrm{d}\sigma(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(x_0) - v(x) \, \mathrm{d}\sigma(x).$$

By assumption this integrand is always nonnegative. We will show that if there is a some $0 < r < R$ and a point $x_1 \in B(x_0, r)$ such that $v(x_1) < v(x_0)$ then the right hand side above is strictly positive. This is a contradiction, and so it follows that $v(x) = v(x_0)$ for all points $x \in B(x_0, R)$.

If there is such a point x_1, then there is an open neighbourhood $U \ni x_1$ such that for all $x \in U$ it holds that $v(x) \leq \frac{1}{2}(v(x_0) + v(x_1))$. Then

$$\begin{aligned}
\frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(x_0) - v(x) \, \mathrm{d}\sigma(x) &\geq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)\cap U} v(x_0) - v(x) \, \mathrm{d}\sigma(x) \\
&\geq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)\cap U} \frac{1}{2}(v(x_0) - v(x_1)) \, \mathrm{d}\sigma(x) \\
&\geq \frac{\sigma(\partial B(x_0,r) \cap U)}{n\omega_n r^{n-1}} \frac{1}{2}(v(x_0) - v(x_1)) \\
&> 0,
\end{aligned}$$

because $\partial B(x_0, r) \cap U$ is an open subset of the sphere and therefore has nonzero measure.

To summarise, we have now shown that v is constant and equal to the maximum value $v(x_0)$ on the largest ball $B(x_0, R)$ that is contained in Ω. Repeating this argument we can show that v is also the same constant on any ball that overlaps $B(x_0, R)$, and so on. Because Ω is connected, this shows that $v(x) = v(x_0)$ on all of Ω.

(c) **(i)** This follows by direct computation

$$\begin{aligned}
\Delta\|\nabla u\|^2 &= \sum_{j,k=1}^{n} \partial_j^2 \left[(\partial_k u)^2\right] = \sum_{j,k=1}^{n} \partial_j \left[2\partial_k u \, \partial_j \partial_k u\right] = 2 \sum_{j,k=1}^{n} \partial_j \partial_k u \, \partial_j \partial_k u + \partial_k u \, \partial_j^2 \partial_k u \\
&= 2 \sum_{j,k=1}^{n} (\partial_j \partial_k u)^2 + 2 \sum_{k=1}^{n} \partial_k u \, \partial_k \Delta u \geq 0.
\end{aligned}$$

(ii) A smooth convex function has the property that $f'' \geq 0$. By the chain rule then

$$\begin{aligned}
\Delta(f \circ u) &= \sum_{j=1}^{n} \partial_j \partial_j (f \circ u) = \sum_{j=1}^{n} \partial_j (f' \circ u) \partial_j u = \sum_{j=1}^{n} (f'' \circ u)(\partial_j u)^2 + (f' \circ u) \partial_j^2 u \\
&= (f'' \circ u)\|\nabla u\|^2 + (f' \circ u)\Delta u \geq 0
\end{aligned}$$

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.