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Introduction to Partial Differential Equations
Exercise sheet 10

2nd December, 2020

30. The heat kernel on $\mathbb{S}^1$.

(See also Exercise 4.22 in the lecture script) Denote the fundamental solution of the heat equation by $\Phi(x,t)$.

(a) Give the definition of a Schwartz function. *(1 Point(s))*

(b) Show that $f(x) = e^{-x^2}(e^{-x^2} + \sin^2 x)$ is a positive Schwartz function whose square root is not a Schwartz function. *(2 Points + 3 Bonus Points)*

(c) Show that for every $t > 0$ the function $\Phi(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is Schwartz function. *(3 Point(s))*

(d) Calculate the Fourier transform of $\Phi(\cdot, t)$ for any $t > 0$. [FYI. $\int_{\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$.] *(3 Point(s))*

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Show that

$$\tilde{f}(x) = \sum_{n \in \mathbb{Z}} f(x+n)$$

defines a smooth periodic function with period 1 (i.e. $\tilde{f}(x+1) = \tilde{f}(x)$). *(2 Point(s))*

(f) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, with period 1, and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ a solution to the heat equation with initial condition $u(x,0) = h(x)$. Show that u remains periodic in the spatial coordinate for all time. *(2 Point(s))*

(g) Conclude that

$$u(x,t) := \int_{\mathbb{S}^1} h(y) \sum_{n \in \mathbb{Z}} \Phi(x - y + n, t).$$

solves the heat equation with the initial condition. *(2 Point(s))*

(h) Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

Show, with the aid of this equality, the relation

$$\Theta(x - y, 4\pi i t) = \sum_{n \in \mathbb{Z}} \Phi(x - y + n, t),$$

where the left hand side is the Jacobi's theta function from Section 4.7. *(3 Bonus Point(s))*

(i) How would you modify Definition 4.14 to give give an abstract definition of the heat kernel $H_{\mathbb{S}^1}$? *(3 Bonus Point(s))*

Solution.

(a) A Schwartz function is a smooth functions whose partial derivatives (of all orders) decay faster than the reciprocal of any polynomial. In other words for all multi-indices α and $k \in \mathbb{N}$

$$\lim_{|x| \to \infty} |x|^k \partial^\alpha f(x) = 0.$$

Clearly because this decays to zero, $\sup |x|^k |\partial^\alpha f(x)|$ exists. Conversely, if this supremum exists, then

$$\lim_{|x|\to\infty} |x|^k |\partial^\alpha f(x)| = \lim_{|x|\to\infty} |x|^{-1} |x|^{k+1} |\partial^\alpha f(x)| \le \lim_{|x|\to\infty} |x|^{-1} \sup |x|^{k+1} |\partial^\alpha f(x)| = 0.$$

Therefore these two conditions are equivalent. The second version, $\sup |x|^k |\partial^\alpha f(x)| < \infty$, is often more useful.

(b) This is a function of one variable, so we do not need to use multi-indices.

$$\begin{aligned}
f &= e^{-2x^2} + e^{-x^2} \sin^2 x, \\
f(k\pi) &= e^{-2\pi^2 k^2}, \\
f' &= -4xe^{-2x^2} - 2xe^{-x^2} \sin^2 x + 2e^{-x^2} \sin x \cos x \\
f'(k\pi) &= -4k\pi e^{-2\pi^2 k^2} \\
f'' &= -4e^{-2x^2} + 16x^2 e^{-2x^2} + 4x^2 e^{-x^2} \sin^2 x - 4xe^{-x^2} \sin x \cos x + 2e^{-x^2} (\cos^2 x - \sin^2 x) \\
f''(k\pi) &= (16\pi^2 k^2 - 4) e^{-2\pi^2 k^2} + 2e^{-\pi^2 k^2}
\end{aligned}$$

The exponential terms are dominant for large $|x|$, so f' decays faster than any polynomial. For higher derivatives it is easy to see that every derivative has the form $e^{-Ax^2} P(x, \sin x, \cos x)$ where $P(x, y, z)$ is a polynomial and $A = 1, 2$. Thus f is Schwartz. Let $g = \sqrt{f}$, then $g' = 0.5 f'/\sqrt{f}$ and $g'' = 0.25(2ff'' - (f')^2)/f^{1.5}$. At the points $x = k\pi$,

$$\begin{aligned}
\frac{2ff'' - (f')^2}{f^{1.5}} &= \frac{2e^{-2\pi^2 k^2}(ae^{-2\pi^2 k^2} + 2e^{-\pi^2 k^2}) - (-4k\pi e^{-2\pi^2 k^2})^2}{(e^{-2\pi^2 k^2})^{1.5}} \\
&= \frac{2ae^{-4\pi^2 k^2} + 4e^{-3\pi^2 k^2} - 16k^2 \pi^2 e^{-4\pi^2 k^2}}{e^{-3\pi^2 k^2}} \\
&\to 4 \text{ for } k \to \infty
\end{aligned}$$

So $|x^{-1} g''(x)|$ will be unbounded for large x.

(c) Fix $t > 0$. Recall the fundamental solution of the heat equation is

$$\Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp -\frac{x_1^2 + \cdots + x_n^2}{4t},$$

which has the form $p(x) \exp -\frac{|x|^2}{4t}$ for some polynomial $p(x)$. Anything of this form also has a derivative of this form

$$\begin{aligned}
\partial_i \left(p(x) \exp -\frac{|x|^2}{4t} \right) &= \partial_i p \exp -\frac{|x|^2}{4t} - \frac{2x_i}{4t} p \exp -\frac{|x|^2}{4t} \\
&= \left(\partial_i p - \frac{x_i}{2t} p \right) \exp -\frac{|x|^2}{4t}.
\end{aligned}$$

And all function of this form decay faster than any polynomial. By induction, Φ is Schwartz.

(d)

$$\begin{aligned}
\hat{\Phi}(k,t) &:= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y}\Phi(y,t)\,dy \\
&= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\left(-2\pi i k\cdot y - \frac{|y|^2}{4t}\right)\,dy \\
&= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\frac{-1}{4t}\left((4t\pi i|k|)^2 + 8t\pi i k\cdot y + |y|^2 - (4t\pi i|k|)^2\right)\,dy \\
&= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\frac{-1}{4t}\left(|4t\pi i k + y|^2 - (4t\pi i|k|)^2\right)\,dy \\
&= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\frac{-1}{4t}\left(|z|^2 + 16t^2\pi^2|k|^2\right)\,dz \\
&= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp(-4t\pi^2|k|^2)\,\exp(-|w|^2)\,(4t)^{n/2}\,dw \\
&= \frac{\exp(-4t\pi^2|k|^2)}{\pi^{n/2}}\int_{\mathbb{R}} \exp(-w_1^2)\,dw_1\cdots\int_{\mathbb{R}} \exp(-w_n^2)\,dw_n \\
&= \frac{\exp(-4t\pi^2|k|^2)}{\pi^{n/2}}\sqrt{\pi}^n \\
&= \exp(-4t\pi^2|k|^2).
\end{aligned}$$

(e) The function is period with period 1 by renaming the summation variable. The difficulty is proving that it converges. But this follows by comparison to the series $\frac{1}{n^2}$. We know that there is a constant C with $|f| < \frac{C}{x^2}$ for all x. Then for $x \in [0,1]$ we have

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |f(x+n)| &\le |f(x)| + |f(x-1)| + C\sum_{n\neq 0,-1}\frac{1}{(x+n)^2} \\
&\le |f(x)| + |f(x-1)| + C\sum_{n\neq 0}\frac{1}{n^2} \\
&\le 2\sup|f| + C\frac{\pi^2}{6},
\end{aligned}$$

which demonstrates uniform convergence. The derivative of a Schwartz function is a Schwartz function, so the same result applies to the sum of the derivatives and shows $\tilde{f}$ is differentiable. Repeating the argument with $\partial_i \tilde{f}$ proves $\tilde{f}$ is smooth.

(f) Consider the difference of $v(x,t) = u(x+1,t) - u(x,t)$. This solves the heat equation with the initial condition $v(x,0) = h(x+1) - h(x) = 0$. Then we use Theorem 4.12 for this PDE to conclude that v must be zero.

(g) The initial condition of the heat equation on the circle is given by $h : [0,1] \to \mathbb{R}$ with $h(0) = h(1)$. Thus we can extend h to a period-1 function on all of $\mathbb{R}$.

By the previous questions, we know that $\tilde{\Phi}$ is a well defined smooth and periodic function and that we can pass an integral through the summation:

$$\begin{aligned}\int_{\mathbb{S}^1} h(y) \sum_{n\in\mathbb{Z}} \Phi(x-y+n,t)\,dy &= \sum_{n\in\mathbb{Z}} \int_0^1 h(y)\Phi(x-(y-n),t)\,dy \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} h(z+n)\Phi(x-z,t)\,dz \\ &= \int_{\mathbb{R}} h(z)\Phi(x-z,t)\,dz.\end{aligned}$$

We know from Theorem 4.3 that this solves the heat equation.

(h) One could try to make an argument using a uniqueness of the heat kernel but we will proceed with the hint suggested. Using the previous parts of this question

$$\begin{aligned}\tilde{\Phi}(z,t) &= \sum \hat{\Phi}(n,t)e^{2\pi inz} \\ &= \sum \exp(-4t\pi^2 n^2)e^{2\pi inz} \\ &= \sum \exp(2\pi inz + \pi i(4\pi it)n^2) \\ &= \Theta(z, 4\pi it)\end{aligned}$$

(i) Let read Definition 4.14 and see what does and does not apply on the circle. We can't use the idea $\mathbb{S}^1 = \{x^2+y^2=1\} \subset \mathbb{R}^2$, because it is not open. And otherwise $\mathbb{S}^1$ is not a subset of $\mathbb{R}^n$. But we can overcome this by using periodic functions on $\mathbb{R}$. "The heat kernel $H_{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^+ \to \mathbb{R}$ is a function on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$ that is periodic in the first two variables with periods 1."

Condition (i) is trivially true because the circle is already closed. So we can omit this condition.

For condition (ii), we have the problem that Φ is not periodic. However we can replace this with $\tilde{\Phi}$ which is periodic. So $(y,t) \mapsto H_{\mathbb{S}^1}(x,y,t) - \tilde{\Phi}(x-y,t)$ should solve the homogeneous heat equation and extend continuously to 0 on $(y,t) \in \mathbb{S}^1 \times \{0\}$.

Note then that $H_{\mathbb{S}^1}(x,y,t) = \tilde{\Phi}(x-y,t)$ itself meets this definition. This is exactly parallel to the way that $\Phi(x-y)$ is the Green's function of the Laplacian on $\mathbb{R}^n$ and $\Phi(x-y,t)$ is the heat kernel on $\mathbb{R}^n$.

31. The connection between the fundamental solutions of the heat equation and the Laplace equation.

Let Φ_H be the fundamental solution of the heat equation and Φ_L the fundamental solution of the Laplace equation on $\mathbb{R}^n$ for $n \geq 3$. Denote the Laplace transformation of Φ_H by

$$G(x,\lambda) := \int_0^\infty \Phi_H(x,t)e^{-\lambda t}\,dt.$$

Show for any fixed $x \in \mathbb{R}^n \setminus \{0\}$ that $g(x) := \lim_{\lambda\to 0} G(x,\lambda)$ exists. Show that g has the form $a\Phi_L + b$ for constants $a, b \in \mathbb{R}$. *(5 Point(s))*

Solution. The limit is

$$g(x) = \frac{1}{(4\pi)^{n/2}} \lim_{\lambda\to 0} \int_0^\infty t^{-n/2} \exp\left(-\frac{|x|^2}{4t} - \lambda t\right) dt$$

Consider the integrand for $t < 1$. For $x \neq 0$, as $t \to 0$ the integrand converges to zero. Therefore it is bounded on $[0,1]$. For $t > 1$ the integrand is bounded by $t^{-n/2}$, which is integrable

$$\int_1^\infty t^{-n/2}\, dt = \frac{1}{1-n/2} t^{1-n/2}\Big|_1^\infty = 0 - \frac{1}{1-n/2}$$

for $n \geq 3$. Therefore by the dominated convergence theorem, we can pass the limit through the integral sign:

$$g(x) = \int_0^\infty \Phi_H(x,t) \cdot 1\, dt.$$

We can compute the Laplacian of this function

$$\Delta g(x) = \int_0^\infty \Delta\Phi_H(x,t)\, dt = -\int_0^\infty \partial_t \Phi_H(x,t)\, dt = -\Phi_H(x,\infty) + \Phi_H(x,0) = 0,$$

which shows it to be harmonic. Also notice that it is spherically symmetric because Φ_H is spherically symmetric in x (it only depends on $|x|$). Finally, we know by the working at the beginning of Section 3.1 that any such harmonic function is a linear function of Φ_L.

32. The heat kernel on $[0,1]$.

(Exercise 4.23 from the lecture script)

(a) Show the final step in the calculation of the heat kernel $H_{[0,1]}$:

$$\sum_{k=1}^\infty e^{-\pi^2 k^2 t}(\cos(k\pi(x-y)) - \cos(k\pi(x+y))) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t)$$

(2 Points)

(b) Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z}+1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2. *(1 Point)*

(c) Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth functions $\tilde{f}$ in $\mathcal{A}$:

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(2n+x) - \sum_{n\in\mathbb{Z}} f(2n-x).$$

(2 Points)

(d) Show for any $h \in \mathcal{A}$, that the solutions of the heat equation with initial value h is for all $t > 0$ a smooth function in $\mathcal{A}$. Conclude form this that the following sum has the properties of the Heat kernel of $[0, 1]$:

$$\sum_{n\in\mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n\in\mathbb{Z}} \Phi(x + 2n + y, t).$$

(3 Bonus Points)

(e) Show the relation

$$H_{[0,1]}(x, y, t) = \sum_{n\in\mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n\in\mathbb{Z}} \Phi(x + 2n + y, t),$$

where the left hand side the heat kernel in terms of theta functions as given in the lecture script. *(2 Bonus Points)*

Solution.

(a) We begin with a theta function:

$$\begin{aligned}
\Theta(\tfrac{z}{2}, \pi i t) &= \sum_{k\in\mathbb{Z}} \exp(\pi i k z + (\pi i)^2 t k^2) \\
&= \sum_{k\in\mathbb{Z}} e^{-\pi^2 t k^2} \exp(\pi i k z) \\
&= 1 + \sum_{k=1}^{\infty} e^{-\pi^2 t k^2} \Big[\exp(\pi i k z) + \exp(-\pi i k z) \Big] \\
&= 1 + \sum_{k=1}^{\infty} e^{-\pi^2 t k^2} 2\cos(\pi k z).
\end{aligned}$$

Taking the difference with $z = x - y$ and $z = x + y$ gives the the result.

(b) The functions of $\mathcal{A}$ are clearly periodic with period 2, since for $n = 2$ we have the relation $f(2 + x) = f(x)$. For $n = 1$ we have that $f(1 + x) = -f(1 - x) = -f(-(1 + x))$, which shows that these functions are odd. Therefore it vanishes at 0 and all even integers. Setting $x = 0$ also gives $f(1 + 0) = -f(1 - 0)$ showing it vanishes at 1, and hence all odd integers also.

Now take any odd function f with period 2. Then clearly we have $f(n + x) = f(x)$ for any even integer n; this is the definition of 'period 2'. For an odd integer n:

$$f(n + x) = f(1 + x) = -f(-(1 + x)) = -f(-1 - x) = -f(1 - x).$$

This shows $f \in \mathcal{A}$.

(c) The proof of uniform convergence is very similar to a previous question. By shifting the summation indices we see that it has period 2 also. It remains to show that it is an odd function

$$\tilde{f}(-x) = \sum_{n\in\mathbb{Z}} f(2n - x) - \sum_{n\in\mathbb{Z}} f(2n + x) = -\tilde{f}(x).$$

(d) We have to check that it fits Definition 4.14 with $\Omega = (0,1)$. For positive time the fundamental solution is smooth and Schwartz. Thus the sums are smooth too and defined on all of $x, y \in \mathbb{R}$. Thus there is no problem extending to the boundary. We need to check that it vanishes on $\partial\Omega = \{0,1\}$. For $y = 0$ this is clear. For $y = 1$, relabel the second index $n = m - 1$ to see the two sums are also equal.

Now we need to verify condition (ii). Consider

$$H_{[0,1]}(x,y,t) - \Phi(x-y,t) = \sum_{n\in\mathbb{Z}, n\neq 0} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t).$$

We know that $\Phi(z,t)$ extends continuously with value 0 as $t \to 0$ if $z \neq 0$ (at $(0,0)$ there is a singularity). If $x \in \Omega = (0,1)$ and $y \in \overline{\Omega} = [0,1]$ then $x - y \in (-1,1)$ and so $x + 2n - y \neq 0$ for $n \neq 0$. Similarly $x + y \in (0,2)$ so the singularities are avoided in the second sum too.

(e) We reuse the result

$$\Theta(z, 4\pi i t) = \sum_{n\in\mathbb{Z}} \Phi(z+n,t).$$

Namely

$$\begin{aligned}
\Theta(\tfrac{1}{2}z, \pi i t) &= \sum_{n\in\mathbb{Z}} \Phi(\tfrac{1}{2}z + n, \tfrac{1}{4}t), \\
\Phi(\tfrac{1}{2}z + n, \tfrac{1}{4}t) &= \frac{1}{(4\pi t)^{n/2}} \exp -\frac{\left|\frac{1}{2}z + n\right|^2}{4 \cdot \frac{1}{4}t} \\
&= \frac{1}{(4\pi t)^{n/2}} \exp -\frac{|z+2n|^2}{4t} \\
&= \Phi(z+2n,t), \\
\Theta(\tfrac{1}{2}z, \pi i t) &= \sum_{n\in\mathbb{Z}} \Phi(z+2n,t).
\end{aligned}$$

Equality now follows.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.