Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 9

25th November, 2020

27. Solutions of the homogeneous heat equation.

Let $u : \mathbb{R}^n \times (0\,\infty) \to \mathbb{R}$ be a solution of the homogeneous heat equation, i.e. $\dot{u} - \Delta u = 0$.

(a) Show for every $\lambda \in \mathbb{R}$ that $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ is a solution to the heat equation. *(2 Point(s))*

(b) Show that $v(x,t) := x \cdot \nabla u + 2t\,\dot{u}$ is also a solution. *(3 Point(s))*

(c) In the situation of $n = 1$, one spatial coordinate, let $v : \mathbb{R} \to \mathbb{R}$ be a given smooth function and $u(x,t) := v(t^{-1}x^2)$. Show that v is a solution of the differential equation

$$4z\,v''(z) + (2+z)\,v'(z) = 0 \quad \text{for } z > 0$$

exactly when u satisfies the heat equation. *(3 Point(s))*

Solution.

(a) This follows because $\partial_t(u(\lambda x, \lambda^2 t)) = \lambda^2 \dot{u}(\lambda x, \lambda^2 t)$ and $\partial_j^2(u(\lambda x, \lambda^2 t)) = \lambda^2(\partial_j^2 u)(\lambda x, \lambda^2 t)$.

(b) The clever approach to this question is to observe that $v = \partial_\lambda u_\lambda|_{\lambda=1}$ must be a solution because u_λ is a solution for all λ.

A direct approach also works. The derivatives with respect to the space and time coordinates are:

$$\partial_t v = x \cdot \nabla\dot{u} + 2\dot{u} + 2t\,\ddot{u} \qquad \partial_j^2 v = 2\partial_j^2 u + x \cdot \nabla(\partial_j^2 u) + 2t\,\partial_j^2 \dot{u},$$

so the Laplacian is $\Delta v = 2\Delta u + x \cdot \nabla(\Delta u) + 2t\,\Delta\dot{u}$. Everything then cancels as required.

(c) Set $z = t^{-1}x^2$. By direct computation again

$$\begin{aligned}
\partial_t &= (\partial_t z)\partial_z = -t^{-2}x^2\partial_z = -t^{-1}z\partial_z \\
\partial_x &= (\partial_x z)\partial_z = 2t^{-1}x\partial_z \\
\partial_x^2 &= 2t^{-1}\partial_z + (2t^{-1}x)^2\partial_z^2 = 2t^{-1}\partial_z + 4t^{-1}z\partial_z^2.
\end{aligned}$$

The result follows since $(\partial_t - \partial_x^2)u = 0$ if and only if $-t^{-1}(zv' + 2v' + 4zv'') = 0$.

28. An alternative description of the solutions of the heat equation.

Suppose that we are given an open region $\Omega \subset \mathbb{R}$ and an infinitely differentiable function $f : \Omega \to \mathbb{R}$. Suppose moreover that there is a constant $M > 0$ with

$$|(\Delta^k f)(x)| \le M^k,$$

for all $x \in \Omega$ and $k \ge 0$. Here Δ^k is the Laplace operator applied k-times. For example, $\Delta^2 f = \Delta(\Delta f)$. By convention, we set $\Delta^0 f = f$.

Show now that

$$u(x,t) := (e^{t\Delta})f(x) := \sum_{k=0}^{\infty} \frac{1}{k!}(\Delta^k f)(x)t^k$$

defines an infinitely differentiable function on $\Omega \times \mathbb{R}$ and further that it solves the initial value problem

$$\dot{u} - \Delta u = 0, \qquad u(x,0) = f(x).$$

(5 Point(s))

Solution. By the assumption, the sum is absolutely convergent for all $(x,t) \in \Omega \times \mathbb{R}$:

$$\sum_{k=0}^{\infty} \left| \frac{1}{k!}(\Delta^k f)(x)t^k \right| \le \sum_{k=0}^{\infty} \frac{1}{k!}(Mt)^k = e^{Mt} < \infty$$

and thus $u(x,t)$ is well-defined. One set of criteria that permits us to differentiate u term-wise is that a series converges at a point and the sum of the derivatives converge uniformly. For the term-wise t-derivatives we have

$$\sum_{k=0}^{\infty} \left| \frac{k \dots (k-l+1)}{k!}(\Delta^k f)(x)t^{k-l} \right| = \sum_{k=0}^{\infty} \left| \frac{1}{k!}(\Delta^{k+l} f)(x)t^k \right| \le \sum_{k=0}^{\infty} \frac{1}{k!} M^l (Mt)^k = M^l e^{Mt} < \infty,$$

which demonstrates uniform convergence in t on compact time intervals, so we may indeed calculate $\dot{u}$ term-wise.

It it not so clear that the function is smooth in x however. One idea I had was to use the mean value theorem for multi-variable functions, which gives

$$|\partial_i f(b) - \partial_i f(a)| = |\partial_i^2 f(c)||b-a|$$

for some $c \in \{ta + (1-t)b \mid t \in (0,1)\}$ where a and b only differ in the i^{th} coordinate. This shows how you might try to bound the odd-order derivatives in terms of even-order derivatives. But I don't see how to bound this by the Laplacian of f. I'm offering a can of soft drink as a prize to any student who can prove the smoothness result or provide a counterexample.

Even without this, it is still possible to show that it is a solution to heat equation however, because one can prove a version of the term-wise derivative formula directly for the linear operator:

$$g \mapsto \lim_{h \to 0} \sum_{i=0}^{n} \frac{g(x+he_i) - 2g(x) + g(x-he_i)}{h^2},$$

which is equal to the Laplacian when the Laplacian exists. Observe then that

$$\sum_{k=0}^{\infty} \left| \frac{1}{k!}(\Delta^{k+l} f)t^k \right| \le \sum_{k=0}^{\infty} \frac{1}{k!} M^{k+l}|t|^k = M^l e^{Mt} < \infty$$

shows the term-wise Δ-derivative is uniformly convergent for $x \in \Omega$. Hence we may freely pass these derivatives through the sum:

$$\dot{u} - \Delta u = \sum_{k=0}^{\infty} \frac{k}{k!}(\Delta^k f)(x)t^{k-1} - \sum_{k=0}^{\infty} \frac{1}{k!}(\Delta^{k+1} f)(x)t^k = 0.$$

Further, $u(x,0) = \frac{1}{0!}\Delta^0 f(x) = f(x)$.

29. An alternative estimate for Corollary 3.4.

First let $\Omega' \subset \mathbb{R}^n \times \mathbb{R}$ be an open and connected region. A function $v : \Omega' \to \mathbb{R}$ is called a *sub-solution* of the heat equation if $\dot{v} - \Delta v \leq 0$.

(a) *Mean value estimate for sub-solutions* Take any point $(x,t) \in \Omega'$ and a small radius $r > 0$ so that $E(x,t,r) \in \Omega'$ (refer to Definition 4.6). Modify the proof the mean value property of the heat equation to show that

$$v(x,t) \leq \frac{1}{4r^n} \int_{E(x,t,r)} v(y,s) \frac{|x-y|^2}{|t-s|^2}\, d^n y\, ds$$

holds for all sub-solutions. *(4 Point(s))*

Now let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and path connected region. We denote the parabolic cylinder of Ω by $\Omega_T := \Omega \times (0,T]$ as in Section 4.4. Suppose that $v : \Omega_T \to \mathbb{R}$ is a sub-solution that extends continuously to $\overline{\Omega_T}$.

(b) *Maximum principle for sub-solutions* Following on from (a), establish that if v takes the value $\sup_{\Omega_T} v$ on Ω_T, then it is constant. *(4 Point(s))*

(c) *A monotonicity property* For $j \in \{1,2\}$ let $f_j : \Omega \times (0,T) \to \mathbb{R}$, $h_j : \Omega \to \mathbb{R}$, and $g_j : \partial\Omega \times [0,T]$ be smooth functions, and likewise let $u_j : \Omega \times (0,T)$ be smooth functions with continuous extensions to the boundary that satisfy

$$\begin{cases} \dot{u}_j - \Delta u_j = f_j & \text{on } \Omega \times (0,T) \\ u_j(x,0) = h_j(x) & \text{on } \Omega \\ u_j = g_j & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

Suppose further that $f_1 \leq f_2$, $g_1 \leq g_2$, and $h_1 \leq h_2$. Show in this case that $u_1 \leq u_2$ as well. *(4 Point(s))*

Solution.

(a) This question is a combination of the idea behind sub-harmonic functions and the mean-value property for solutions of the heat equation. The proof of this part closely follows the proof of Theorem 4.7 in the lecture script. We give the proof only at the point $(x,t) = (0,0)$; it follows at other points by translation of the integral. Define

$$\phi(r) = \frac{1}{r^n} \int\limits_{E(0,0,r)} v(y,s) \frac{|y|^2}{s^2}\, dy\, ds$$

to be weighted average of v on $E(0,0,r)$, viewed as a function of r. Its derivative is shown to be

$$\phi'(r) = \frac{1}{r^{n+1}} \int\limits_{E(0,0,r)} -4n\dot{v}\psi + 4\left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) y \cdot \nabla v\, dy\, ds.$$

At this step we use the assumption that $\dot{v} \leq \Delta v$. This then shows that $\phi'(r) \geq 0$, so ϕ is an increasing function of r. The script also contains a proof that $\lim_{r\to 0} \phi(r) = 4v(0,0)$. Finally then $v(0,0) = \frac{1}{4}\phi(0) \leq \frac{1}{4}\phi(r)$ completes this question.

(b) Suppose that $M = v(x_0, t_0)$ is the maximum of v on Ω_T. For any r such that $E(x_0, t_0, r)$ is contained in Ω_T, the mean value property implies

$$M \leq \frac{1}{4r^n} \int_{E(x,t,r)} v(y,s) \frac{|x-y|^2}{|t-s|^2} \, d^n y \, ds \leq M.$$

By an argument that we've seen before, if there were any point of $E(x_0, t_0, r)$ where $v \neq M$ then (because of the continuity of v) we could take a small ball $B \subset E$ around this point where $v < M - \delta$ for some $\delta > 0$. It would then follow that

$$\begin{aligned}
M &= \frac{1}{4r^n} \int\limits_{E(0,0,r)} v(y,s) \frac{|y|^2}{s^2} \, dy \, ds \\
&= \frac{1}{4r^n} \left(\int\limits_{E \setminus B} + \int\limits_{B} \right) v(y,s) \frac{|y|^2}{s^2} \, dy \, ds \\
&\leq \frac{1}{4r^n} \int\limits_{E \setminus B} M \frac{|y|^2}{s^2} \, dy \, ds + \frac{1}{4r^n} \int\limits_{B} (M - \delta) \frac{|y|^2}{s^2} \, dy \, ds \\
&= M \frac{1}{4r^n} \int\limits_{E} \frac{|y|^2}{s^2} \, dy \, ds - \delta \frac{1}{4r^n} \int\limits_{B} \frac{|y|^2}{s^2} \, dy \, ds \\
&= M - \delta \frac{1}{4r^n} \int\limits_{B} \frac{|y|^2}{s^2} \, dy \, ds < M,
\end{aligned}$$

which is a contradiction. Thus it must be that v is constant on $E(x_0, t_0, r)$. This can be extended to any other points in Ω_T by taking a path between the maximum and any point, covering the path by (finitely many) sets of the form $E(x, t, r)$ and applying the argument on each set.

There is a small detail here that we should note, namely that (x_0, t_0) is on the boundary of $E(x_0, t_0, r)$. In particular every other point lies in the past: if $(y, s) \in E(x_0, t_0, r)$ and $s \geq t_0$ then $(y, s) = (x_0, t_0)$. Therefore this argument also applies to points in $\Omega \times \{T\}$, which is part of the boundary of Ω_T. This is different than for sub-harmonic functions and motivates the definition of Ω_T which includes the points (x, T). The argument does not apply to other points of the boundary, because there is no r such that $E(x, t, r)$ is contained in the domain.

(c) This is also very familiar. Set $v = u_1 - u_2$. Because $(\partial_t - \Delta)v = f_1 - f_2 \leq 0$, we know that v is a sub-solution. From the boundary data, we also know that v is non-positive on Ω and $\partial\Omega \times [0, T]$. We know that if maximum of v occurs on Ω_T, then v is constant and so evalutation at a known boundary point shows that it is a non-positive constant. If it is non-constant, then the maximum of v on $\overline{\Omega_T}$ occurs on Ω or $\partial\Omega \times [0, T]$ by part (b). Hence the function is bounded from above by a non-positive number.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.