Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 2

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3. Royale with Cheese

Recall Burgers' equation from Example 1.6 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $u_0(x) = x$.

(a) According to Theorem 1.5, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point(s))*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 point(s))*

(c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution. *(2 point(s))*

(d) Finally, derive the following solution to the initial value problem:

$$u(x, t) = \frac{x}{1+t}.$$

(2 point(s))

(e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 point(s))*

(f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point(s))*

Solution.

(a) The condition in the theorem depends on the bound $f''(u_0(x))u_0'(x) \geq -\alpha$. For this equation, both $f''(u) = 1$ and $u_0'(x) = 1$, so the product is bound below easily by $\alpha = 0$. Hence the theorem says that the unique solution exists for all time.

(b) By the chain rule,

$$\frac{d}{ds}u(x(s), t(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{dt}{ds}\dot{u} + \frac{dx}{ds}\partial_x u.$$

If we compare this to the PDE, then we see that we should choose $t' = 1$, i.e. $t = s$, and $x' = u$.

(c) The characteristics are the rays $x = x_0 t + x_0$ for $x_0 \in \mathbb{R}$. The solution takes the value x_0 on the corresponding ray. We can see that the 'mass' (the conserved quantity) is flowing away from the origin.

(d) For the chosen characteristics we know that $\frac{d}{dt}u(x(t), t) = 0$ by the choice of the constraints on the path. Hence u is constant on this path and equal to $u_0(x(0))$. The ODE $x'(s) = u(x(s), s) = u_0(x(0))$ is now easily to integrate, giving

$$x(t) = x_0 + u_0(x_0)t = x_0 + x_0 t = x_0(1 + t).$$

In other words, we know that for all x_0 and t

$$u(x(t), t) = u(x_0(1 + t), t) = u_0(x_0) = x_0.$$

We have now found the solution, but to make it clearer, let $\tilde{x} = x_0(1 + t)$. Then

$$u(\tilde{x}, t) = \frac{\tilde{x}}{1 + t}.$$

(e) The solution is clearly well defined for $t > 0$ and all x. We compute

$$\dot{u} = -\frac{x}{(1 + t)^2}, \qquad \partial_x u = \frac{1}{1 + t}.$$

Thus we see that the partial differential equation is solved. As too is the initial value $u(x, 0) = x = u_0(x)$.

(f) Because such PDEs resemble the chain rule. Hence we can identity the derivatives $x_i'(s)$ of the path with the coefficient functions in PDE and reduce it to a system of ODEs.

4. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.7. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x, t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x, t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x, t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x, t)-plane for this function. *(2 point(s))*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(3 point(s))*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(3 point(s))*

Solution.

(a) Refer to lecture script.

(b) The characteristics for $t < 1$ are described in the lecture script

$$x = \begin{cases} t + x_0 & \text{for } x_0 \leq 0 \\ t(1 - x_0) + x_0 & \text{for } 0 < x_0 \leq 1 \\ x_0 & \text{for } 1 < x_0. \end{cases}$$

However, for $t > 1$ there are two regions: $x > 1$ and $x < 1$. In the former the characteristics continue to be horizontal lines. In the lower region they are lines with gradient 1.

(c) The discontinuity is for $t > 1$ when the solution jumps from 0 to 1. This occurs on the line $y(t) = 1$. Hence $\dot{y} = 0$. On the other hand, $f(u) = \frac{1}{2}u^2$, the value of u on the upper side of the discontinuity is $v^r(1^+, t) = 0$ and $v^l(1^-, t) = 1$. The right hand side of the Rankine-Hugonoit condition is then

$$\frac{\frac{1}{2} \cdot 0^2 - \frac{1}{2} \cdot 1^2}{0 - 1} = \frac{1}{2}.$$

This shows that v does not fulfil the condition.

(d) We know that mass is conserved away from the discontinuity. Therefore we jus need to know how much is being lost across the discontinuity. This is easy to compute in this case because the discontinuity is not moving, $y(t) = 1$. So first we compute the amount of mass in some interval containing the discontinuity, say $x \in [0, 2]$, for $t > 1$:

$$\int_0^2 v(x, t)\, dx = \int_0^1 1\, dx + \int_1^2 0\, dx = 1.$$

So there is a constant amount of mass in the interval. And then we compute how much mass is moving in and out of this interval:

$$f(u(0, t)) - f(u(2, t)) = f(1) - f(0) = \frac{1}{2}.$$

So there is a constant inflow of 0.5 units of mass per unit of time. Hence the system must be losing 0.5, because this inflow is not increasing the amount in the interval.

5. You're not in traffic, you are traffic

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) Argue that $f(u) = s_m u(1 - u/u_m)$ is a reasonable model. Hence write down a PDE to describe the traffic flow. *(2 point(s))*

(b) Consider the situation of a traffic light at $x = 0$: to the left of the traffic light, the cars are queued up at maximum density. To the right of the traffic light, the road is empty. Now, at time $t = 0$, the traffic light turns green. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(6 point(s))*

Solution.

(a) The density flux of the cars should be the density of the cars multiplied by the speed they are travelling $f = us$. We already know that speed depends on the car density u, being zero for $u = u_m$ and s_m for $u = 0$. Assuming a linear relationship gives $s(u) = s_m(1 - u/u_m)$.

Now, cars are a conserved quantity; have you ever seen a car vanish? Therefore they it is reasonable to use the conservation PDE model. Differentiating f gives

$$\dot{u} + s_m \left(1 - 2\frac{u}{u_m}\right) \partial_x u = 0.$$

(b) Choose units so that $s_m = 1$ and $u_m = 2$. The PDE is now

$$\dot{u} + (1 - u)\, \partial_x u = 0.$$

The characteristics are $x = x_0 + (1 - u_0(x_0))t$, in other words

$$\begin{cases} x = x_0 - t & \text{for } x_0 < 0 \\ x = x_0 + t & \text{for } x_0 > 0. \end{cases}$$

Physically we can explain this as there being a region $x > t$ where the first car at the light, now driving full-speed, has not yet reached and another region $x < -t$ where the traffic is still completely packed and the cars cannot move. In between cars can move with some speed, but density prevents them from moving at full speed.

This middle region is not determined by the initial conditions, so there is possibility to have many solutions. If there was a jump between maximum density and no cars, then the Rankine-Hugonoit condition would say it would have a slope of

$$\dot{y} = \frac{u_m(1 - u_m/u_m) - 0(1 - 0/u_m)}{u_m - 0} = 0.$$

The interpretation is that the lights turn green and nobody moves. This is consistent with the equations if you say that the cars at the front are also in a maximum density region.

But we prefer solutions that are as regular as possible (and also drivers who drive when the light is green, honk honk). Again, by characteristics, if $u(x,t)$ is C^1 in this region then it must be constant on lines through the origin: $x = ct$ for $c \in [-1,1]$. It must therefore be equal to some function $g(x/t) = g(c)$ with $g(-1) = 2$ and $g(1) = 0$. The PDE then reduces to an ODE.

$$\begin{aligned} -\frac{x}{t^2}g' + (1-g)\cdot\frac{1}{t}g' &= 0 \\ -cg' + (1-g)g' &= 0 \\ g'\cdot(-c+1-g) &= 0. \end{aligned}$$

So either $g(c)$ is constant, which contradicts the endpoint conditions, or $g(c) = 1 - c$. In summary

$$u(x,t) = \begin{cases} 2 & \text{for } x < -t \\ 1 - \frac{x}{t} & \text{for } -t \le x \le t \\ 0 & \text{for } t < x \end{cases}$$

is a continuous solution.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.