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Introduction to Partial Differential Equations
Exercise sheet 7

11th November, 2020

21. Hackneyed Harnack.

Let $n \geq 3$, $r > 0$ and $B(0,r)$ the open ball in $\mathbb{R}^n$. Further, let $u : B(0,r) \to \mathbb{R}$ be a harmonic function with $u \geq 0$. Show that the following inequality holds for all $x \in B(0,r)$:

$$r^{n-2} \frac{r - |x|}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r+|x|}{(r-|x|)^{n-1}} u(0).$$

(5 Point(s))

Solution. Consider the special form of the Poisson Representation formula for harmonic functions on $B(0,r)$

$$u(x) = \frac{r^2 - |x|^2}{nr\omega_n} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} \, \mathrm{d}\sigma(y).$$

The least value of $|x-y|$ for $y \in \partial B(0,r)$ is $r - |x|$. Together with the mean value property this gives

$$u(x) \leq \frac{r^2 - |x|^2}{nr\omega_n} \frac{1}{(r-|x|)^n} \times n\omega_n r^{n-1} u(0),$$

namely the desired upper bound. For the lower bound, note that the greatest value of $|x-y|$ for $y \in \partial B(0,r)$ is $r + |x|$.

22. Harmonic Polynomials.

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. A *real homogeneous polynomial of degree d* is a linear combination of monomials of the form $Q = x_1^{d_1} \dots x_n^{d_n}$ with $d_k \in \mathbb{N}_0$ and $d_1 + \cdots + d_n = d$. The vector space of real homogeneous polynomials of degree d is denoted $\mathcal{P}(d,n)$. We want to determine the dimension of the subspace of harmonic polynomials

$$\mathcal{H}(d,n) := \{P \in \mathcal{P}(d,n) \mid \Delta P = 0\}.$$

(a) Show using combinatorics that $\dim \mathcal{P}(d,n) = \binom{n+d-1}{d}$. *(2 Point(s))*

(b) Show that the Laplacian of a homogeneous polynomial of degree d is either zero or a homogeneous polynomial of degree $d-2$. *(2 Point(s))*

(c) Show that the linear map $\Delta : \mathcal{P}(d+2,n) \to \mathcal{P}(d,n)$ is surjective.
Hint: It suffices to show that every monomial is in the image of Δ. One may prove this by induction on n and d. *(5 Point(s))*

(d) What is the dimension of $\mathcal{H}(d,n)$? *(1 Point(s))*

Solution.

(a) This is a standard combinatorics argument. Represent an integer sum $d_1 + \cdots + d_n = d$ as a sequence of counters $*$ and separators $|$. For example $1+0+3 = 4$ is the sequence $*||***$. We see that the sequence uses d counters and $n-1$ separators. The number of sequences, and therefore the number of monomials, is $\binom{d+n-1}{d}$. The monomials span $\mathcal{P}(d,n)$ so this is also its dimension.

(b) Observe that $\partial_j^2 x_1^{d_1}\dots x_n^{d_n} = d_j(d_j-1)x_1^{d_1}\dots x_j^{d_j-2}\dots x_n^{d_n}$ is either zero or a homogeneous polynomial of degree $d-2$. The Laplacian of a monomial is the sum of such terms, and therefore also either zero or a homogeneous polynomial of degree $d-2$. By linearity of the Laplacian we have the desired result.

(c) Let us state the proposition thusly: For every $n \geq 1$ and $d \geq 0$, every monomial in $\mathcal{P}(d,n)$ lies in the image of Δ.

We follow the suggestion of the hint. For $n = 1$ we have have that $\mathcal{P}(d,1) = \mathbb{R}x^d$ and the Laplacian is simply the second derivative:

$$\Delta \frac{1}{(d+1)(d+2)} x^{d+2} = \frac{d^2}{dx^2} \frac{1}{(d+1)(d+2)} x^{d+2} = x^d.$$

Therefore the proposition is true for $n = 1$ and every d. For the purposes of induction, suppose that it is true for all $n \leq N$ and all d.

We now show by a second induction on d that it is true for $\mathcal{P}(d, N+1)$. There is only one monomial of degree 0, namely $1 = x_1^0 \dots x_n^0$ and this is $\Delta(\frac{1}{2}x_1^2)$. Suppose then that it is true for all $d \leq D$. Choose any monomial P in $\mathcal{P}(D+1, N+1)$. Write $P = Qx_{n+1}^e$ for $Q \in \mathcal{P}(D+1-e, N)$. By assumption, there exists $\tilde{Q} \in \mathcal{P}(D+3-e, N)$ with $\Delta\tilde{Q} = Q$ on $\mathbb{R}^N$ but clearly also on $\mathbb{R}^{n+1}$ because $\tilde{Q}$ does not depend on x_{n+1}. Consider

$$\Delta(\tilde{Q}x_{n+1}^e) = P + e(e-1)\tilde{Q}x_{n+1}^{e-2}.$$

The term $e(e-1)\tilde{Q}x_{n+1}^{e-2}$ is either zero or lies in $\mathcal{P}(D-1, N+1)$ and so is equal to ΔR for $R = 0$ or $R \in \mathcal{P}(D+, N+1)$. Hence $P = \Delta(\tilde{Q}x_{n+1}^e - R)$, showing that P lies in the image of Δ.

Thus by the induction on d we have shown that the proposition holds for all $d \geq 0$ for $n = N+1$. This then completes the induction step in n, and so the proposition is true in general.

(d) For $d = 0, 1$, every homogeneous polynomial is harmonic, so the dimension is just 1 and n respectively. For $d \geq 2$, we characterise $\mathcal{H}(d,n)$ as the kernel of $\Delta : \mathcal{P}(d,n) \to \mathcal{P}(d-2,n)$. But we know that this is a surjective linear map, and we know the dimensions of the domain and codomain, so the dimension of the kernel is the difference: $\binom{n+d-1}{d} - \binom{n+d-3}{d-2}$.

23. Never judge a book by its cover.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(5 Point(s))*

Solution. Let $v = u_2 - u_1$. This is a harmonic function and on the boundary it is equal to $g_2 - g_1 \geq 0$. If v is negative at any point, then this would contradict the maximum principle by being lower than the boundary. Hence $v \geq 0$, which is exactly that $u_1 \leq u_2$.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.