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Introduction to Partial Differential Equations
Exercise sheet 5

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General Comment: Distributions are defined in Definition 2.6 as *continuous* linear maps. Recall that a linear map is continuous if and only if it is a bounded linear operator ($\Leftrightarrow \overline{F(B(0,1))}$ is compact). But the topology is a little complicated to describe in this situation, so please just use the condition given in the definition directly.

13. The Delta Quadrant.

(a) Show that

$$F : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_{\mathbb{R}} x^3 \cdot \phi''(x)\ \mathrm{d}x$$

is a distribution on $\mathbb{R}$, and define a function $f : \mathbb{R} \to \mathbb{R}$ with

$$F(\phi) = \int_{\mathbb{R}} f(x) \cdot \phi(x)\ \mathrm{d}x \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$

(4 Point(s))

(b) Show that the Dirac-Distribution

$$\delta : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \phi(0)$$

is indeed a distribution on $\mathbb{R}$ and prove that there does *not* exist a function $g : \mathbb{R} \to \mathbb{R}$ with

$$\delta(\phi) = \int_{\mathbb{R}} g(x) \cdot \phi(x)\ \mathrm{d}x \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$

(2+4 Point(s))

(c) Calculate the derivatives F' and δ' of the distributions in parts (a) und (b) respectively.

(2+2 Point(s))

Solution.

(a) We need to show that F is a linear map $F : C_0^\infty \to \mathbb{R}$ that is continuous with respect to the semi-norms $\|\cdot\|_{\mathbb{K},\alpha}$. The main obstacle is that $x^3 \notin L^1(\mathbb{R})$, otherwise the continuity is immediate. So choose any compact K. It contains a furthest point x_m from the origin: $|x_m| = R$. Then for any $\phi \in C_0^\infty(K)$,

$$|f(\phi)| \leq \int_K |x^3||\phi''|\ \mathrm{d}x \leq R^3 \int_K |\phi''|\ \mathrm{d}x \leq R^3 \int_K \sup_{y \in K} |\phi''(y)|\ \mathrm{d}x = R^3 \mu(K) \|\phi\|_{K,(2)}.$$

This shows that F is continuous. It is also linear; for constants $a, b \in \mathbb{R}$

$$F(a\phi + b\psi) = \int_{\mathbb{R}} x^3 (a\phi + b\psi)''\ \mathrm{d}x = \int_{\mathbb{R}} x^3 (a\phi'' + b\psi'')\ \mathrm{d}x = aF(\phi) + bF(\psi).$$

Not only is F a distribution, it is actually one of the special distributions that comes from a function. To see this, apply integration by parts twice to an interval $[a, b] \supset \operatorname{supp} \phi$:

$$\int_{\mathbb{R}} x^3 \phi''(x)\ \mathrm{d}x = 0 - \int_{\mathbb{R}} 3x^2 \phi'(x)\ \mathrm{d}x = -0 + \int_{\mathbb{R}} 6x\, \phi(x)\ \mathrm{d}x,$$

so the sought after $f(x)$ is $6x$.

(b) The addition and scaling of functions is defined pointwise, so linearity of this functional follows by definition. For continuity, choose any compact set K and consider test functions $\phi \in C_0^\infty(K)$. We have

$$|\delta(\phi)| = |\phi(0)| \leq \sup_{x\in K} |\phi(x)| = \|\phi\|_{K,(0)}.$$

It remains to show that this distribution is not induced by a function. To this end, consider the standard bump function $\psi(x) = A\exp(|x|^2 - 1)^{-1}$ for $|x| < 1$ and identically zero otherwise. We choose the constant A so that the integral $\int_{\mathbb{R}} \psi = 1$. The associated family of functions $\psi_\epsilon = \epsilon^{-1}\psi(\epsilon^{-1}x)$ have the properties $\operatorname{supp}\psi_\epsilon \subseteq [-\epsilon, \epsilon]$, $\int_{\mathbb{R}} \psi_\epsilon = 1$, and $\psi_\epsilon(0) = \epsilon^{-1}$.

Suppose now that the distribution δ was induced by a function g. We compute

$$\epsilon^{-1} = \delta(\psi_\epsilon) = \int_{-\epsilon}^{\epsilon} g(x)\psi_\epsilon(x)\,\mathrm{d}x \leq \int_{-\epsilon}^{\epsilon} (\sup_{y\in[-\epsilon,\epsilon]} g(y))\psi_\epsilon(x)\,\mathrm{d}x = \sup_{y\in[-\epsilon,\epsilon]} g(y).$$

But the limit as $\epsilon \to 0$ of the right side is $g(0)$ and the limit of the left side is ∞. Thus g cannot be a function on $\mathbb{R}$.

We can also see how the functional applies to $\psi_\epsilon(x - x_0)$ for $x_0 \neq 0$. For small enough ϵ, $\delta(\psi_\epsilon(x - x_0)) = 0$ because 0 then lies outside its support. By the same caluclation as above then $0 = \sup_{y\in[x_0-\epsilon,x_0+\epsilon]}$ and in the limit as $\epsilon \to 0$ this implies $g(x_0) = 0$. Therefore we see that $g(x) = 0$ almost everywhere. But then g does not have the same property as δ.

(c) The definition of the derivative of a distribution is $F'(\phi) = -F(\phi')$. Therefore

$$F'(\phi) = -\int_{\mathbb{R}} x^3\phi'''(x)\,\mathrm{d}x = \int_{\mathbb{R}} 6\phi(x)\,\mathrm{d}x.$$

This demonstrates that as F is induced by $6x$, so too is F' induced by 6. In this sense the derivative of distributions is a generalisation of the normal derivative.

We apply the same definition to δ:

$$\delta'(\phi) = -\delta(\phi') = -\phi'(0).$$

14. An induced distribution.

Let $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ and $\psi \in C_0^\infty(\mathbb{R}^m)$. Define

$$G : C_0^\infty(\mathbb{R}^n) \to \mathbb{R},$$
$$\varphi \mapsto F(\varphi \times \psi).$$

Show that G is a Distribution on $C_0^\infty(\mathbb{R}^n)$, i.e. $G \in D'(\mathbb{R}^n)$.
(*Caution:* Don't forget to show, that G is well-defined.) *(5 Point(s))*

Solution. For $\psi \in C_0^\infty(\mathbb{R}^m)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, note that $\psi\varphi$ is again a smooth function, and its support it contain in the Cartesian product of the supports of ψ and φ. So we can indeed apply F to the product. As to continuity of G, let $L = \operatorname{supp}\psi \subset \mathbb{R}^m$ and choose a compact set $K \subset \mathbb{R}^n$. For any function $\varphi \in C_0^\infty(K)$, the norm estimate for F gives

$$|G(\varphi)| = |F(\varphi\psi)| \leq C_1\|\varphi\psi\|_{K\times L,\alpha_1} + \cdots + C_M\|\varphi\psi\|_{K\times L,\alpha_M}.$$

We can also decompose the norms like so

$$\begin{aligned}
\|\varphi\psi\|_{K\times L,\alpha} &= \sup_{(x,y)\in K\times L} |\partial^\alpha(\varphi(x)\psi(y))| \\
&= \sup_{(x,y)\in K\times L} \left|\partial^{\alpha'}\varphi(x)\partial^{\alpha''}\psi(y))\right| \\
&\leq \sup_{x\in K} |\partial^{\alpha'}\varphi(x)| \sup_{y\in L} |\partial^{\alpha''}\psi(y))| \\
&= \|\varphi\|_{K,\alpha'}\, \|\psi\|_{L,\alpha''},
\end{aligned}$$

where $\alpha = (\alpha', \alpha'') \in \mathbb{N}_0^{n+m} = \mathbb{N}_0^n \times \mathbb{N}_0^m$ is a decomposition of the multiindex. In this situation, the norms of ψ are fixed constants, so this is a linear combination of the norms of ϕ on K. Substitution of these estimates into the prior bound of $|G(\varphi)|$ gives a bound of the required form.

15. The Crucial Kernel.

When a the partial derivative of a function is zero, it is constant in that direction. In this question we investigate what it means when a distribution has a derivative that is zero. Let $F \in D'(\mathbb{R}^n \times \mathbb{R})$ and let (x, t) with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denote the elements in $\mathbb{R}^n \times \mathbb{R}$.
We want to show that: $\partial_t F = 0$ if and only if there is a distribution $G \in D'(\mathbb{R}^n)$ such that

$$F(\varphi) = G\bigg(\int_{\mathbb{R}} \varphi(-,t)dt\bigg).$$

From a certain point of view then, F does not depend on the t coordinate. In order to show the statement prove the following steps. First, define

$$\begin{aligned}
\mathcal{I} : \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) &\to \mathcal{D}(\mathbb{R}^n), \\
\varphi &\mapsto \bigg(x \mapsto \int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t\bigg).
\end{aligned}$$

- **(a)** (Optional) Show, that $\mathcal{I}$ is continuous and linear. *(3 Point(s))*
- **(b)** Show that a function $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ belongs to the kernel of $\mathcal{I}$ if and only if it is the t-derivative of another such function. *(3 Point(s))*
- **(c)** Show that for $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, $\partial_t F = 0$ if and only if $F \equiv 0$ on the kernel of $\mathcal{I}$. *(2 Point(s))*
- **(d)** Finally show the statement by showing that $\partial_t F = 0$ if and only if there exists a $G \in \mathcal{D}'(\mathbb{R}^n)$ with $F(\varphi) = G(\mathcal{I}(\varphi))$. *(2 Point(s))*

Solution.

(a) Linearity follows from linearity of the integral, but perhaps it is good to establish notations:

$$\mathcal{I}(a\varphi + b\psi)(x) = a\int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t + b\int_{-\infty}^{\infty} \psi(x,t)\ \mathrm{d}t = a\mathcal{I}(\varphi)(x) + b\mathcal{I}(\psi)(x).$$

The function $\mathcal{I}(\varphi)$ is also smooth, because we may pass derivatives through the integral sign.

The question of continuity depends on which norms are being used, and is more subtle. Recall that a linear function is continuous if and only if it is a bounded operator. This explains Definition 2.6. It is enough therefore to bound $\mathcal{I}(\varphi)$ with respect to all of the semi-norms on $\mathcal{D}(\mathbb{R}^n)$. Fix any compact sets $K \subset \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n \times \mathbb{R}$, and choose $\varphi \in C_0^\infty(L)$.

$$\|\mathcal{I}(\varphi)\|_{K,\alpha} = \sup_{x\in K}\left|\partial^\alpha \int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t\right| \leq \sup_{x\in K}\int_{-\infty}^{\infty} |\partial^\alpha\varphi(x,t)|\ \mathrm{d}t.$$

Now, we don't need to integrate from $-\infty$ to ∞ because φ has compact support. By projecting L to $\mathbb{R}$, we see that there is a bound $T \in \mathbb{R}$ such that if $|t| > T$ then $\varphi(x,t) = 0$ for all $x \in \mathbb{R}^n$.

$$\sup_{x\in K}\int_{-\infty}^{\infty} |\partial^\alpha\varphi(x,t)|\ \mathrm{d}t = \sup_{x\in K}\int_{-T}^{T} |\partial^\alpha\varphi(x,t)|\ \mathrm{d}t \leq 2T\sup_{x\in K}\sup_{t\in[-T,T]}|\partial^\alpha\varphi(x,t)| \leq 2T\|\varphi\|_{L,\alpha}.$$

This shows that $\mathcal{I}$ is a bounded linear operator and therefore is continuous.

(b) Firstly, what does it mean for φ to be in the kernel of $\mathcal{I}$? It means for all $x \in \mathbb{R}^n$

$$\int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t = 0.$$

Suppose then that φ is in the kernel of $\mathcal{I}$. We must show that there exists $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ such that $\varphi = \partial_t\psi$. Define

$$\psi(x,t) = \int_{-\infty}^{t} \varphi(x,t)\ \mathrm{d}t.$$

This is a smooth function and its derivative is φ, so it remains to show that it has compact support. As we saw in the previous part, there exists a bound T such that for all $|t| > T$ the function $\varphi(x,t) = 0$ for any $x \in \mathbb{R}^n$. Thus $\psi(x,t) = 0$ for $t < -T$ and $\psi(x,t)$ is a constant for $t > T$. However, the assumption that φ is in the kernel of $\mathcal{I}$ tells us that this constant is zero. Thus ψ also has compact support.

Conversely, take any $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Note that

$$\int_{-\infty}^{\infty} \partial_t\psi\ \mathrm{d}t = \psi\Big|_{t=-\infty}^{t=\infty} = 0,$$

so that $\partial_t\psi$ is in the kernel of $\mathcal{I}$.

(c) F is a distribution and $\partial_t F$ means the distributional derivative, ie $\partial_t F(\varphi) = -F(\partial_t\varphi)$. Suppose that $\partial_t F = 0$ and that φ is in the kernel of $\mathcal{I}$. From part (b), we know that $\varphi = \partial_t\psi$ for some $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Therefore we apply $\partial_t F$ to ψ to conclude

$$0 = \partial_t F(\psi) = -F(\varphi).$$

This shows that F vanishes on the kernel of $\mathcal{I}$.

In the other direction, suppose that F vanishes on the kernel of $\mathcal{I}$ and take any $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Again by part (b), we know that $\partial_t \psi$ is in the kernel of $\mathcal{I}$. Therefore

$$\partial_t F(\psi) = -F(\partial_t \psi) = 0.$$

(d) Before we address F, note that $\mathcal{I}$ is surjective. Explicitly, if $\omega : \mathbb{R} \to \mathbb{R}$ is a function with compact support and $\int_{\mathbb{R}} \omega(t)\, \mathrm{d}t = 1$, and g is any function in $\mathcal{D}(\mathbb{R}^n)$ then $\mathcal{I}(g(x)\omega(t)) = g$. Therefore, as topological vector spaces, $\mathcal{D}(\mathbb{R}^n \times \mathbb{R})/\ker \mathcal{I}$ and $\mathcal{D}(\mathbb{R}^n)$ are isomorphic.

If $\partial_t F = 0$, from the part (c) we know that F vanishes on $\ker \mathcal{I}$ and so this isomorphism induces a well defined map $G \in \mathcal{D}'(\mathbb{R}^n)$ such that $F(\varphi) = G(\mathcal{I}(\varphi))$. The reverse is immediate: if $F(\varphi) = G(\mathcal{I}(\varphi))$ then F vanishes on the kernel of $\mathcal{I}$ and so must have $\partial_t F = 0$.

16. You can now write "Transport-Distribution Expert" on your résumé.

In this exercise we show that there is a one-to-one correspondence between distributions solving the linear transport equation and distributions describing the corresponding initial values g.

(a) Show that for any distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$, the following distribution solves the equation $\partial_t \tilde{F} = 0$:

$$\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \text{ with } \tilde{F}(\phi) = F(\tilde{\phi}) \text{ and } \tilde{\phi}(y,t) = \phi(y - bt, t) \text{ for all } (y,t) \in \mathbb{R}^n \times \mathbb{R}.$$

(2 Point(s))

(b) Show that for any mollifier $(\lambda_\epsilon)_{\epsilon>0}$ on $\mathbb{R}$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$ the functions

$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \text{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$

belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. *(1 Point(s))*

(c) Recall $\mathcal{I}$ from the *The Crucial Kernel*. Let $\tilde{F} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solve the equation $\partial_t \tilde{F} = 0$. We have already proved that there exists a distribution $G \in \mathcal{D}(\mathbb{R}^n)$, such that $\tilde{F}(\phi) = G(\mathcal{I}(\phi))$. Argue therefore that $\tilde{F}(\phi \times \lambda_\epsilon)$ does not depend on $\epsilon > 0$. *(1 Point(s))*

(d) Show that for any $G \in \mathcal{D}(\mathbb{R}^n)$ the following $F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solves $(\partial_t + b\nabla)F = 0$:

$$F : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}, \qquad \phi \mapsto G\left(\int_{\mathbb{R}} \mathrm{T}_{-tb}\phi(\cdot, t)\, \mathrm{d}t\right),$$

where T_{-tb} is a translation operator. *(3 Point(s))*

(e) Show that $G \to F$ is bijective onto $\{F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \mid (\partial_t + b\nabla)F = 0\}$. *(3 Point(s))*

Solution.

(a) The core of this question is how does the chain rule of differentiation look for distributions? The order of operations is a a little subtle, so to be clear let us write the translation operator $T(\phi) = \phi(y - bt, t)$ explicitly. In other words, $\tilde{F}(\phi) = F(T\phi)$. First observe that T commutes with the spatial derivatives:

$$\partial_k \tilde{\phi} = \partial_k(T\phi) = \partial_k(\phi(x - bt, t)) = T(\partial_k \phi).$$

On the other hand, T does *not* commute with the time derivative. By the chain rule,

$$\partial_t \tilde{\phi} = \partial_t(T\phi) = T(\tilde{\nabla}\phi) \cdot \partial_t T = \begin{pmatrix} T(\nabla\phi) \\ T(\partial_t \phi) \end{pmatrix} \cdot \begin{pmatrix} -b \\ 1 \end{pmatrix} = -b \cdot T(\nabla\phi) + T(\partial_t \phi),$$

where $\tilde{\nabla}$ is the gradient with respect to $\mathbb{R}^n \times \mathbb{R}$ and ∇ is the gradient with respect to $\mathbb{R}^n$. Together this says that

$$T(\partial_t \phi) = \partial_t(T\phi) + b \cdot T(\nabla\phi) = \partial_t(T\phi) + b \cdot \nabla(T\phi).$$

Now we are in a position where we can address the question. In the following we use the definition of the derivative of a distribution and the definition of $\tilde{F}$ and pay close attention to the order of operators:

$$\begin{aligned} \partial_t \tilde{F}(\phi) &= -\tilde{F}(\partial_t \phi) = -F(T(\partial_t \phi)) = -F(\partial_t(T\phi) + b \cdot \nabla(T\phi)) \\ &= -F(\partial_t(T\phi)) - F(b \cdot \nabla(T\phi)) = \partial_t F(T\phi) + b \cdot \nabla F(T\phi) \\ &= (\partial_t + b \cdot \nabla) F(T\phi) = 0. \end{aligned}$$

(b) The product of two smooth functions is smooth. So it only remains to show that the product has compact support. Let K is the support of ϕ and the support of λ_ϵ is I. If $(x, t) \notin K \times I$ then either $\phi(x) = 0$ or $\lambda_\epsilon(t) = 0$ (or both). In both cases the product is zero. This shows that the support of the product is contained in $K \times I$, which is a bounded set, and thus the support of the product must be compact.

(c) We compute

$$\mathcal{I}(\phi \times \lambda_\epsilon)(x) = \int_{-\infty}^{\infty} \phi(x)\lambda_\epsilon(t)\,\mathrm{d}t = \phi(x) \int_{-\infty}^{\infty} \lambda_\epsilon(t)\,\mathrm{d}t = \phi(x),$$

because the integral of a mollifier is always 1. In other words, $\mathcal{I}(\phi \times \lambda_\epsilon) = \phi$. As explained the question, the condition that $\partial_t \tilde{F} = 0$ means that it is of the form $\tilde{F}(\psi) = G(\mathcal{I}(\psi))$ for some distribution G. Therefore $\tilde{F}(\phi \times \lambda_\epsilon) = G(\mathcal{I}(\phi \times \lambda_\epsilon)) = G(\phi)$ is independent of ϵ.

(d) Again, the order of operators in this question is somewhat subtle. Let us introduce a translation $S(\phi) = \phi(x + bt, t)$. This is similar to T from part (a), in fact they are inverses, and we have that S commutes with ∇ but

$$S(\partial_t \phi) = \partial_t(S\phi) - b \cdot S(\nabla\phi).$$

One could also write the integral part of this formula using the operator $\mathcal{I}$, but we don't have to interchange its position, so we will leave it as an integral so as not to be more abstract than necessary. Perhaps it would be a good exercise to rewrite the following proof using $\mathcal{I}$.

In this notation we have that

$$F(\phi) := G\left(x \mapsto \int_{\mathbb{R}} S\phi \,\mathrm{d}t\right).$$

Let us compute the t-derivative of this F: for any test function ϕ,

$$\begin{aligned}
\partial_t F(\phi) &= -F(\partial_t \phi) = -G\left(x \mapsto \int_{\mathbb{R}} S(\partial_t \phi)\,\mathrm{d}t\right) \\
&= -G\left(x \mapsto \int_{\mathbb{R}} \partial_t(S\phi) - b \cdot S(\nabla\phi)\,\mathrm{d}t\right) \\
&= -G\left(x \mapsto 0 - \int_{\mathbb{R}} b \cdot S(\nabla\phi)\,\mathrm{d}t\right) \\
&= \sum_{k=1}^{n} b_k G\left(x \mapsto \int_{\mathbb{R}} S(\partial_k \phi)\,\mathrm{d}t\right) \\
&= \sum_{k=1}^{n} b_k F(\partial_k \phi) = -b \cdot \nabla F(\phi).
\end{aligned}$$

This shows that it solves the transport equation.

(e) Part (d) shows that the mapping $G \mapsto F$ is well-defined. Suppose then that we had a solution F of the transport equation. Part (a) shows there is an associated distribution $\tilde{F}$ with the property that $\partial_t \tilde{F} = 0$. Using part (c) we have $\tilde{F}(\phi) = G(\mathcal{I}(\phi))$ for some $G \in \mathcal{D}'(\mathbb{R}^n)$. This gives a mapping $F \mapsto G$.

It remains to show that these mappings are inverse to one another, but observe

$$F(\phi) = F(TS\phi) = \tilde{F}(S\phi) = G(\mathcal{I}(S\phi)),$$

which crucially relies on T and S being inverse translations. The mapping is therefore bijective.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.