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Introduction to Partial Differential Equations
Exercise sheet 11

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33. Scaling the heat kernel.

Find the heat kernel of

(a) $\mathbb{R}/c\mathbb{R}$ with $c > 0$, *(3 Point(s))*

(b) $[a, b]$ with $-\infty < a < b < \infty$. *(3 Point(s))*

Solution.

(a) To begin, think about what functions there are between $\mathbb{R}/c\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$. The obvious one is $x \mapsto x/c$. How does the fundamental solution for $n = 1$ behave under scaling then?

$$\Phi(cx, c^2t) = \frac{1}{\sqrt{4\pi c^2 t}} \exp\left(-\frac{(cx)^2}{4c^2t}\right) = \frac{1}{c}\Phi(x, t).$$

So the fundamental solution on $\mathbb{R}/c\mathbb{Z}$ is $\frac{1}{c}\tilde{\Phi}(z/c, t/c^2)$. From a previous exercise (30(i)), this must be the heat kernel.

(b) In a similar way, one transformation from $[a, b]$ to $[0, 1]$ is $\frac{x-a}{b-a}$. We already know how the fundamental solution behaves under scaling, but it does not simplify when shifted. However, remember that the shifting behaviour is built into the idea of the heat kernel/Greens function, namely $G(x, y) = \phi(x - y)$. If we set $H_{[a,b]}(x, y, t) = \frac{1}{b-a} H_{[0,1]}\left(\frac{x-a}{b-a}, \frac{y-1}{b-a}, \frac{t}{(b-a)^2}\right)$ then clearly it vanishes at the boundaries $x = a, b$. Further

$$\begin{aligned}
&H_{[a,b]}(x, y, t) - \Phi(x - y, t) \\
&\quad = \frac{1}{b-a}\left[H_{[0,1]}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right) - \Phi\left(\frac{x-y}{b-a}, \frac{t}{(b-a)^2}\right)\right] \\
&\quad = \frac{1}{b-a}\left[H_{[0,1]}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right) - \Phi\left(\frac{x-a}{b-a} - \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right)\right] \\
&\quad = \frac{1}{b-a}\left[H_{[0,1]}\left(x', y', t'\right) - \Phi\left(x', y', t'\right)\right],
\end{aligned}$$

so this must solve the homogenous heat equation and vanish initially ($t = 0 \Leftrightarrow t' = 0$). Therefore it is the heat kernel.

34. Some like it hot.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases}
\dot{u} - 7\partial_{xx}u = 0 & \text{for } x \in (0, \pi),\ t > 0 \\
u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\
u(x, 0) = 3\sin(2x) - 6\sin(5x) & \text{for } x \in (0, \pi).
\end{cases}$$

(6 Point(s))

Solution. Firstly, this isn't quite the heat equation, but we can rescale time to absorb the factor 7, namely $s = 7t$. We have just calculated the heat kernel for an interval, so let's use it! By Theorem 4.16

$$\begin{aligned}
u(x,s) &= 0 - 0 + \int_{[0,\pi]} [3\sin(2y) - 6\sin(5y)]\, H_{[0,\pi]}(x,y,s)\,\mathrm{d}y \\
&= 0 - 0 + \int_{[0,\pi]} [3\sin(2y) - 6\sin(5y)]\, \frac{1}{\pi} H_{[0,1]}\left(\tfrac{x}{\pi}, \tfrac{y}{\pi}, \tfrac{s}{\pi^2}\right)\,\mathrm{d}y \\
&= \int_0^\pi [3\sin(2y) - 6\sin(5y)]\, \frac{1}{\pi} \sum_{k=1}^\infty e^{-k^2 s} 2\sin(kx)\sin(ky)\,\mathrm{d}y \\
&= \frac{2}{\pi} \sum_{k=1}^\infty e^{-k^2 s}\sin(kx) \left[\int_0^\pi 3\sin(2y)\sin(ky)\,\mathrm{d}y - \int_0^\pi 6\sin(5y)\sin(ky)\,\mathrm{d}y\right].
\end{aligned}$$

Know it is relatively easy to see, by applying integration by parts twice, that $\int_0^\pi \sin mz \sin nz$ is zero if $m \neq n$ and is $\pi/2$ if they are equal. Thus all but two terms of the sum are zero.

$$\begin{aligned}
u(x,s) &= \frac{2}{\pi} e^{-4s}\sin(2x)\, 3\frac{\pi}{2} - \frac{2}{\pi} e^{-25s}\sin(5x)\, 6\frac{\pi}{2} \\
&= 3e^{-4s}\sin(2x) - 6e^{-25s}\sin(5x) \\
u(x,t) &= 3e^{-28t}\sin(2x) - 6e^{-175t}\sin(5x)
\end{aligned}$$

The temperature falls very quickly, so make sure you have a jacket. We can also check our solution:

$$\begin{aligned}
\dot{u}(x,t) &= -28 \cdot 3e^{-28t}\sin(2x) + 175 \cdot 6e^{-175t}\sin(5x) \\
\partial_{xx} u(x,t) &= -4 \cdot 3e^{-28t}\sin(2x) + 25 \cdot 6e^{-175t}\sin(5x).
\end{aligned}$$

Because this is just the 1-dimensional heat equation, there are a variety of other effective methods: separation of variables to reduce it to two ODES, or a Laplace transform to reduce it to an inhomogeneous ODE are two that spring to mind.

Another approach to the above integrals would be to write the heat kernel in the form $\hat{\Phi}(n,t)e^{2\pi i n x}$. Then again all but two of the integrals will be zero and we see the solution is the sum of two Fourier transforms of the fundamental solution.

35. Out of the frying pan, into the fire.

Find the solution $u : (0,\pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases}
\dot{u} - \partial_{xx} u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\
u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\
u(x,0) = x^2(\pi - x) & \text{for } x \in (0,\pi).
\end{cases}$$

Further, show that your solution obeys $\displaystyle\int_0^\pi u(x,t)\,\mathrm{d}x = 8 \sum_{k \text{ odd}} \frac{1}{k^4} e^{-k^2 t}$. *(8 Bonus Point(s))*

Solution. We will again you the heat kernel and the representation formula, except this the solution will not be elementary:

$$\begin{aligned} u(x,t) &= 0 - 0 + \int_{[0,\pi]} y^2(\pi - y)\,\frac{1}{\pi} H_{[0,1]}\left(\tfrac{x}{\pi}, \tfrac{y}{\pi}, \tfrac{t}{\pi^2}\right)\,\mathrm{d}y \\ &= \frac{2}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t}\sin(kx)\int_{[0,\pi]} y^2(\pi - y)\,\sin(ky)\,\mathrm{d}y. \end{aligned}$$

The integral can be computed by repeated integration by parts and is $-2\pi k^{-3}(1 + 2\cos k\pi)$. Hence

$$u(x,t) = -4\sum_{k=1}^{\infty} k^{-3}(1 + 2\cos k\pi)\,e^{-k^2 t}\sin(kx).$$

This is the solution. We can easily compute its value numerically because the series is so rapidly convergent. But we see that it is quite difficult to understand the overall shape. For example, what it the highest temperature of the rod at any given time? Despite the initial condition being positive, it is even not immediately clear that the solution is even positive (though it is, this follows because it is an alternating series and so is bounded between its partial sums). Let's split this into even and odd terms:

$$u(x,t) = 4\sum_{k \text{ odd}}^{\infty} k^{-3}\,e^{-k^2 t}\sin(kx) - 12\sum_{k \text{ even}}^{\infty} k^{-3}\,e^{-k^2 t}\sin(kx).$$

Because $\int_0^\pi \sin kx \;\mathrm{d}x$ is zero if k is even and $2/k$ if k is odd, the result about the total heat in the domain follows. Notice that the total amount of heat is not conserved in this situation, because the ends of the rod are being kept at a constant temperature of zero and so heat is escaping through these ends.

36. Do nothing by halves ... again.

Consider the following heat equation with initial and boundary conditions, on the set $\Omega = (0,\infty)$:

$$\begin{cases} \dot{u} - \Delta u = 0 & \text{on } \Omega \times (0,\infty) \\ u(0,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = h(x) & \text{for } x \in \Omega, \end{cases}$$

where h is a bounded continuous function on $\overline{\Omega}$ with $h(0) = 0$. We are seeking a solution which is $C^2(\overline{\Omega} \times (0,\infty))$ and extends continuously to $\overline{\Omega} \times [0,\infty)$. Note, $C^k(A)$ for a non-open set A means that all derivatives up to an including order k (which are only defined on $\operatorname{int} A$) are continuous and also extend continuously to A.

(a) Suppose that we had such a solution $u(x,t)$. Extend both u and h to functions $\tilde{u} : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ and $\tilde{h} : \mathbb{R} \to \mathbb{R}$ that are odd in x, ie $\tilde{u}(x,t) = -\tilde{u}(-x,t)$ and $\tilde{h}(x) = -\tilde{h}(-x)$. What regularity must $\tilde{u}$ have? *(2 Point(s))*

(b) Suppose now that $\tilde{u}$ has sufficient regularity. Show that it solves the following Cauchy problem:

$$\begin{cases} \dot{v} - \Delta v = 0 & \text{on } \mathbb{R} \times (0, \infty) \\ v(x, 0) = \tilde{h}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

(2 Point(s))

(c) Show that when u is a bounded solution of the PDE on the half-line, that $|u|$ is bounded by the supremum of $|h|$. Explain why it follows that a bounded solution is unique.

(2 Point(s))

(d) Show that

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(1 - e^{-\frac{xy}{t}}\right) \exp\left(-\frac{|x-y|^2}{4t}\right) h(y)\,\mathrm{d}y$$

is the unique bounded solution. *(3 Point(s))*

(e) Prove the estimate

$$|u(x,t)| \le \frac{x}{\sqrt{4\pi}\, t^{3/2}} \int_0^\infty \exp\left(-\frac{|x-y|^2}{4t}\right) y\,|h(y)|\,\mathrm{d}y.$$

Hint: The exponential function is convex, so bounded by its tangent line.

(3 Point(s))

Solution.

(a) The reflected functions have the same regularity away from the origin, so we only need to investigate the regularity at $x = 0$. Both u and h are zero for $x = 0$, so their odd extensions are continuous

$$\lim_{x \to 0^-} \tilde{h}(x) = \lim_{x \to 0^-} -h(-x) = \lim_{x \to 0^+} -h(x) = 0 = \lim_{x \to 0^+} \tilde{h}(x).$$

In general we only assume $h(x)$ is continuous, so let us focus on u.

Suppose $u \in C^2(\overline{\Omega} \times (0, \infty))$ and that it extends continuously to $\overline{\Omega} \times [0, \infty)$. Observe that by the definition of partial derivatives

$$\dot{u}(0,t) = \lim_{h \to 0} \frac{u(0, t+h) - u(0,t)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Next, $\dot{u} \in C^1(\overline{\Omega} \times (0, \infty))$, which means that the limit of $\dot{u}$ as $x \to 0$ exists and (from above) is zero. Hence its odd extension is continuous. The same reasoning shows that $\partial_{tt}\tilde{u}$ is continuous.

For the x derivatives, the first derivative always extends to $x = 0$ continuously because

$$\lim_{x \to 0^-} \frac{\partial \tilde{u}}{\partial x}(x,t) = \lim_{x \to 0^-} \frac{\partial}{\partial x}(-u(-x,t)) = \lim_{x \to 0^-} \frac{\partial u}{\partial x}(-x,t) = \lim_{x \to 0^+} \frac{\partial u}{\partial x}(x,t).$$

Since it follows that $\partial_x u(0,t) = 0$, we apply the same argument as above to see that $\partial_{xt}\tilde{u}$ is continuous on $\mathbb{R} \times (0, \infty)$. Finally, we also know that $\Delta u = \partial_{xx} u \in C(\overline{\Omega} \times (0, \infty))$. From the PDE then, the limit as $x \to 0$ of Δu is the same that of $\dot{u}$ and hence is zero.

In summary $\tilde{u} \in C^2(\mathbb{R} \times (0, \infty))$.

(b) It clearly solves the problem for all $x \neq 0$. We already saw in the previous question that both $\dot{u}$ and $\partial_{xx} u$ are zero on $x = 0$, so their equality is somewhat boring. Likewise $\tilde{u}(0,0) = 0 = \tilde{h}(0)$.

(c) This follows from applying the maximum principle to the reflected function (Theorem 4.11):

$$|u| \leq |\tilde{u}| \leq \sum_{(x,t) \in \mathbb{R} \times (0,\infty)} |\tilde{u}(x,t)| \leq \sum_{x \in \mathbb{R}} |\tilde{h}(x)| = \sum_{x \in \mathbb{R}} |h(x)|.$$

If there were two bounded solutions, their difference would also be a bounded solution with initial condition $h \equiv 0$. But then it follows that the difference must be zero everywhere.

(d) We know the solution for $\tilde{u}$: this is Theorem 4.12,

$$\begin{aligned}
\sqrt{4\pi t}\, u(x,t) &= \sqrt{4\pi t}\, \tilde{u}(x,t) \\
&= \int_{-\infty}^{\infty} \tilde{h}(y) \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y \\
&= \int_{0}^{\infty} h(y) \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y - \int_{-\infty}^{0} h(-y) \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y \\
&= \int_{0}^{\infty} h(y) \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y - \int_{0}^{\infty} h(y) \exp\left(-\frac{(x+y)^2}{4t}\right) \mathrm{d}y \\
&= \int_{0}^{\infty} h(y) \left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right] \mathrm{d}y \\
&= \int_{0}^{\infty} h(y) \left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x-y)^2 + 4xy}{4t}\right)\right] \mathrm{d}y \\
&= \int_{0}^{\infty} h(y) \left[1 - \exp\left(-\frac{xy}{t}\right)\right] \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y
\end{aligned}$$

(e) Following the hint, the tangent line of $w = 1 - e^{-z}$ at $z = 0$ is just $w = z$ and it bounds the function from above. It follows from the last part that

$$\sqrt{4\pi t}\, |u(x,t)| \leq \int_{0}^{\infty} |h(y)| \left[\frac{xy}{t}\right] \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y$$

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.