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Introduction to Partial Differential Equations
Exercise sheet 12

9th December, 2020

37. 1D Waves.

(a) Show that the a smooth function $u = u(\zeta, \eta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution to $\partial_{\zeta\eta} u = 0$ exactly when it is of the form $u(\zeta, \eta) = F(\zeta) + G(\eta)$, for smooth functions $F, G : \mathbb{R} \to \mathbb{R}$.
(2 Point(s))

(b) Under the parameterisation $\zeta = x + t, \eta = x - t$, show that u obeys the one dimensional wave equation $(\partial_{tt} - \partial_{xx})u = 0$ exactly when it solves the PDE in (a). *(2 Point(s))*

(c) From parts (a) and (b), derive D'Alembert's formula. *(2 Point(s))*

Solution.

(a) Suppose u is a solution. Integrating once, we see that $\partial_\eta u = g(\eta)$, because for each value of η, $\partial_\eta u$ must be constant in ζ. Integrating again gives $u = F(\zeta) + \int g(\eta)\, d\eta =: F(\zeta) + G(\eta)$. The converse is immediate.

(b) Using the chain rule we compute the operator under the change of variables. Note $x = \frac{1}{2}(\zeta + \eta)$ and $y = \frac{1}{2}(\zeta - \eta)$.

$$\frac{\partial}{\partial \zeta} = \frac{\partial x}{\partial \zeta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \zeta}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial \eta} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial t}$$
$$\partial_{\zeta\eta} = \frac{1}{4}(\partial_x - \partial t)(\partial_x + \partial t) = \frac{1}{4}(\partial_{xx} - \partial xt + \partial_{tx} - \partial tt) = \frac{1}{4}(\partial_{xx} - \partial tt).$$

We see then that the operator $\partial_{tt} - \partial_{xx}$ is just a rescaling of $\partial_{\zeta\eta}$.

(c) We are asked to solve the problem of Theorem 5.1:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

where g is twice continuously differentiable and f is only once continuously differentiable. From (a), we know that the equation is simpler in the (ζ, η) coordinates, where $u(\zeta, \eta) = F(\zeta) + G(\eta)$ solves the wave equation. The functions F and G are only defined up to a constant between them (ie $u = (F - C) + (G + C)$ also), so without loss of generality choose $G(0) = 0$.

When $t = 0$ that corresponds to $x = \zeta = \eta$. So the initial conditions say $F(\zeta) + G(\zeta) = g(\zeta)$ and $\partial_t u|_{t=0} = (\partial_\zeta - \partial_\eta)u|_{\zeta=\eta} = F'(\zeta) - G'(\zeta) = h(\zeta)$. Integrating the latter gives

$$\int_0^\zeta h(y)\, dy = \int_0^\zeta F'(y) - G'(y)\, dy = F(\zeta) - G(\zeta) - (F(0) - G(0)) = F(\zeta) - G(\zeta) - g(0).$$

Now we have two linear equations for F and G, so solving gives

$$F(\zeta) = \frac{1}{2}\left[g(\zeta) + g(0) + \int_0^\zeta h(y)\, dy\right], \qquad G(\zeta) = \frac{1}{2}\left[g(\zeta) - g(0) - \int_0^\zeta h(y)\, dy\right].$$

Changing the variable in G back to η and summing gives:

$$
\begin{aligned}
u &= F(\zeta) + G(\eta) \\
&= \frac{1}{2}\left[g(\zeta) + \int_0^\zeta h(y)\,dy\right] + \frac{1}{2}\left[g(\eta) - \int_0^\eta h(y)\,dy\right] \\
&= \frac{1}{2}\left[g(\zeta) + g(\eta)\right] + \frac{1}{2}\int_\eta^\zeta h(y)\,dy.
\end{aligned}
$$

Finally, changing back to (x,t) coordinates gives the desired formula.

38. Faster!

How should you modify D'Alembert's formula for this situation?

$$
\begin{cases}
\partial_{tt}u - a^2\partial_{xx}u = 0 \\
u(x,0) = g(x) \\
\partial_t u(x,0) = h(x),
\end{cases}
$$

Solve this for the initial data $a = 2$, $g(x) = \sin(x)$ and $h(x) = 1$. *(2+2 Point(s))*

Solution. One can rescale one of the coordinates to compensate for the factor of a^2. Namely, let $\tau = at$. Because $t = 0$ when $\tau = 0$, the first initial condition is unchanged. The second initial condition however reads $a\partial_\tau u(x,0) = h(x)$. Using the formula for the solution to this new initial value problem for the waver equation, but then further making the substitution $\tau = at$, gives

$$
u(x,t) = \frac{1}{2}\left[g(x+at) + g(x-at)\right] + \frac{1}{2}\int_{x-at}^{x+at} \frac{1}{a}h(y)\,dy.
$$

With the given initial data

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} 1\,dy \\
&= \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + t.
\end{aligned}
$$

39. Weak waves.

Let U be an open set in $\mathbb{R}^n$ and $\Omega = U \times (0,T)$ be a cylinder in $\mathbb{R}^{n+1}$. A continuous function u is called a *weak solution* of the wave equation on Ω if

$$
\int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\,u\,dx\,dt = 0
$$

for every test function $\varphi \in C_0^\infty(\Omega)$. Solutions to the wave equation in the ordinary sense are called *classical* or *strong* in this context.

(a) Show that $u \in C^2(\Omega)$ is a weak solution if and only if it is a classical solution. *(3 Point(s))*

(b) Suppose that $(u_k)_{k\in\mathbb{N}}$ is a sequence of weak solutions that converges to u with local uniform continuity on Ω. Show that u is also a weak solution. *(4 Point(s))*

Solution.

(a) When u is differentiable, that allows us to apply the divergence theorem/integration by parts. Let's split the integral into two terms:

$$\int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u \, dx\, dt = \int_U \int_0^T \partial_{tt}\varphi\, u \, dt\, dx - \int_0^T \int_U \Delta\varphi\, u\, dx\, dt.$$

We apply integration by parts twice to the first part:

$$\int_0^T \partial_{tt}\varphi\, u\, dt = -\int_0^T \partial_t\varphi\, \partial_t u\, dt = \int_0^T \varphi\, \partial_{tt} u\, dt,$$

because $\partial_t\varphi$ and φ belong to $C_0^\infty(\Omega)$ and $t = 0, T$ lie in the boundary of Ω. The second Green's theorem shows that $\int_U \Delta\varphi\, u\, dx = \int_U \varphi\, \Delta u\, dx$:

$$\int_U \Delta\varphi\, u - \varphi\, \Delta u\, dx = \int_{\partial U} (u\nabla\varphi - \varphi\nabla u)\cdot N\, d\sigma = 0.$$

Together this shows

$$\int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u\, dx\, dt = \int_\Omega \varphi(\partial_{tt}u - \Delta u)\, dx\, dt.$$

Clearly this is zero if u is a classical solution. Conversely, if this is zero for all test functions φ, then by the fundamental lemma of the calculus of variations it follows that u is a classical solution.

(b) In general, the adjective 'local' means that the property holds for a neighbourhood of every point. So in this case, for every point $p \in \Omega$ there exists a neighbourhood $p \in \Omega_p \subset \Omega$ where

$$\sup_{y\in\Omega_p} |u(y) - u_k(y)| \to 0$$

as $k \to \infty$.

We know that uniform continuity of a sequence of functions preserves continuity in the limit. Hence $u \in C(\Omega)$. Now take any test function φ. The support of φ is clearly covered by the collection $\{\Omega_p \mid p \in \operatorname{supp}\varphi\}$. And because the support is compact, there is a finite subcover $\{\Omega_{p_i}\}$. We then compute

$$\begin{aligned}
&\left| \int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u\, dx\, dt - \lim_{k\to\infty} \int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u_k\, dx\, dt \right| \\
&\qquad \le \lim_{k\to\infty} \int_\Omega |\partial_{tt}\varphi - \Delta\varphi|\, |u - u_k|\, dx\, dt \\
&\qquad \le \lim_{k\to\infty} \sum_i \int_{\Omega_{p_i}} |\partial_{tt}\varphi - \Delta\varphi|\, |u - u_k|\, dx\, dt \\
&\qquad \le \sum_i \left(\lim_{k\to\infty} \sup_{y\in\Omega_p} |u - u_k| \right) \int_{\Omega_{p_i}} |\partial_{tt}\varphi - \Delta\varphi|\, dx\, dt \\
&\qquad \to 0.
\end{aligned}$$

This shows that when the u_k are weak solutions (and so the corresponding integrals are all zero) then u is also a weak solution.

40. 1D Waves in the weak sense.

(a) Show that for given continuous functions F, G on $\mathbb{R}$, the function $u(x,t) = F(x+t)+G(x-t)$ is a weak solution of the one dimensional wave equation.
[Hint. Mollify F and G.] *(4 Point(s))*

(b) Show that the Fourier series

$$u(x,t) = \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)\, \sin kx,$$

where a_k and b_k are real sequences with $\sum |a_k| + |b_k| < \infty$, is a weak solution of the one dimensional wave equation. *(3 Point(s))*

Solution.

(a) Firstly note that the function u is continuous on the cylinder. The difficulty is that we expect that u is a solution based on its form, but we don't know that it has enough regularity to actually differentiate. One trick to increase regularity is to use a mollifier ϕ_ϵ and set $F_\epsilon = F * \phi_\epsilon, G_\epsilon * \phi_\epsilon$, and $u_\epsilon = F_\epsilon(x+t) + G_\epsilon(x-t)$. The functions u_ϵ are now smooth and strong solutions to the wave equation. From 39(a) we know that u_ϵ is a weak solution if it is a strong solution. Finally taking $\epsilon \to 0$ and using 39(b) shows that u is a weak solution, since mollifications converge locally uniformly.

(b) Consider the terms individually:

$$(a_k \cos kt+b_k \sin kt)\, \sin kx = \frac{1}{2}a_k(\sin k(x+t)+\sin k(x-t))+\frac{1}{2}b_k(-\cos k(x+t)+\cos k(x-t)).$$

This shows each term, and therefore the partial sums, are strong solutions. Hence also weak solutions. The estimate

$$\left|\sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)\, \sin kx\right| \le \sum_{k=1}^{\infty}(|a_k \cos kt| + |b_k \sin kt|)\,|\sin kx| \le \sum_{k=1}^{\infty} |a_k| + |b_k|$$

show that the sum is uniformly convergent. Hence u is a weak solution.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.