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Introduction to Partial Differential Equations
Exercise sheet 1

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1. **Multiindices and the Generalised Leibniz rule.** In this question we introduce multiindex notation. A *multiindex* of n variables is a vector $\gamma \in \mathbb{N}_0^n$.

   (a) Let $x = (x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$. Write out the full expression for the derivative $\partial^{(0,2,1)}$. *(1 point)*

   (b) Why do we need to assume that partial derivatives commute for multiindex notation to be useful? *(1 point)*

   (c) Which multiindices satisfy $|\gamma| \leq 2$ and which satisfy $\gamma \leq (0, 2, 1)$? *(2 points)*

   (d) The generalised binomial coefficient for multiindices is defined to be

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n}.$$

   One justification for calling these binomial coefficients is the following property. Let $e_j = (0, \ldots, 1, \ldots, 0)$ be the multiindex with 1 is the j-th position and 0 in all other positions. Then for any j

$$\binom{\gamma}{\delta} = \binom{\gamma - e_j}{\delta - e_j} + \binom{\gamma - e_j}{\delta}.$$

   Prove this property. *(2 points)*

   (e) Let $u, v : \Omega \to \mathbb{R}$ be smooth enough functions on an open subset $\Omega \subset \mathbb{R}^n$. Show for all multiindices $\gamma \in \mathbb{N}_0^n$ the following product rule:

$$\partial^\gamma(uv) = \sum_{0 \leq \delta \leq \gamma} \binom{\gamma}{\delta} \partial^\delta u \, \partial^{\gamma-\delta} v$$

   (6 points)

Solution.

(a) $\partial^{(0,2,1)} = \partial_1^0 \partial_2^2 \partial_3^1 = \partial_2^2 \partial_3 = \frac{\partial^2}{(\partial x_2)^2} \frac{\partial}{\partial x_3}$. The lower number is the coordinate and the upper number is the order of the derivative. The zeroth order derivative is just the function itself.

(b) The multiindex notation applies the partial derivatives in a certain order. It does not have the capacity to express the same derivatives applied in a different order, for example the difference between $\partial_1\partial_2$ and $\partial_2\partial_1$. So, to be useful, this order must not make a difference, i.e. the partial derivatives must commute.

(c) We compute for $j = 1$

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n} = \left[\binom{\gamma_1 - 1}{\delta_1 - 1} + \binom{\gamma_1 - 1}{\delta_1}\right]\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n} = \binom{\gamma - e_1}{\delta - e_1} + \binom{\gamma - e_1}{\delta},$$

where we have used the similar property of the ordinary binomial coefficients. Clearly it also holds for any other j too.

(d) We prove the statement by induction. First, observe that the statement is trivial for $\gamma = 0$ (the zero multiindex):

$$\partial^0(uv) = uv = \binom{0}{0}\partial^0 u\,\partial^0 v = \sum_{\delta\le 0}\binom{0}{\delta}\partial^\delta u\,\partial^{0-\delta}v.$$

So we now assume that the statement is true for all multiindices γ whose length is k. We must then prove that the statement holds for all multiindices $\gamma + e_j$ where γ is length k. It is clear by renaming the variables and the commutativity of the partial derivatives that it is enough to prove the induction step for e_1. We compute

$$\begin{aligned}
\partial^{\gamma+e_1}(uv) &= \partial_1\partial^\gamma(uv) = \partial_1 \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^\delta u\,\partial^{\gamma-\delta}v = \partial_1\left(\sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^\delta u\,\partial^{\gamma-\delta}v\right)\\
&= \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\left(\partial^{\delta+e_1}u\,\partial^{\gamma-\delta}v + \partial^\delta u\,\partial^{\gamma+e_1-\delta}v\right)\\
&= \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^{\delta+e_1}u\,\partial^{(\gamma+e_1)-(\delta+e_1)}v + \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^\delta u\,\partial^{\gamma+e_1-\delta}v\\
&= \sum_{e_1\le\delta'\le\gamma+e_1}\binom{\gamma}{\delta'-e_1}\partial^{\delta'}u\,\partial^{\gamma+e_1-\delta'}v + \sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\partial^\delta u\,\partial^{\gamma+e_1-\delta}v\\
&= \binom{\gamma}{\gamma}\partial^{\gamma+e_1}u\,\partial^0 v + \sum_{e_1\le\delta\le\gamma}\left[\binom{\gamma}{\delta-e_1}+\binom{\gamma}{\delta}\right]\partial^\delta u\,\partial^{\gamma+e_1-\delta}v\\
&\quad + \sum_{0\le\delta\le\gamma,\,\delta_1=0}\binom{\gamma}{\delta}\partial^\delta u\,\partial^{\gamma+e_1-\delta}v\\
&= \binom{\gamma+e_1}{\gamma+e_1}\partial^{\gamma+e_1}u\,\partial^0 v + \sum_{e_1\le\delta\le\gamma}\binom{\gamma+e_1}{\delta}\partial^\delta u\,\partial^{\gamma+e_1-\delta}v\\
&\quad + \sum_{0\le\delta\le\gamma+e_1,\,\delta_1=0}\binom{\gamma}{\delta}\partial^\delta u\,\partial^{\gamma+e_1-\delta}v\\
&= \sum_{0\le\delta\le\gamma+e_1}\binom{\gamma+e_1}{\delta}\partial^\delta u\,\partial^{\gamma+e_1-\delta}v
\end{aligned}$$

So we have proved the induction step. Hence the formula is true for multiindices γ of any length.

2. Inhomogeneous Transport Equation. First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous equation

$$\frac{du}{dt} = f(t).$$

(a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

(b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique, if you haven't seen the lecture/read the script.) *(2 points)*

Solution.

(a) A solution is given by the integral $\int_0^t f(s)\, ds$.

(b) A solution to the initial value problem is

$$u(t) = c + \int_0^t f(s)\, ds.$$

This is a sum of a solution to the homogeneous equation that satisfies the initial value and a solution to the inhomogeneous equation that has an initial value of zero.

Suppose that there is another solution v. Then by linearity $u - v$ solves the homogeneous equation $\frac{du}{dt} = 0$ and therefore is a constant. But we know these solutions have the same initial value and hence $u - v = 0$ for all time.

(c) We compute the t and x derivatives of the integral term

$$\begin{aligned}
\frac{\partial}{\partial t} \int_0^t f(x + (s - t)b, s)\, ds &= f(x + (t - t)b, t)\, ds + \int_0^t (-b) \cdot \nabla f(x + (s - t)b, s)\, ds \\
&= f(x, t)\, ds - b \cdot \int_0^t \nabla f(x + (s - t)b, s)\, ds
\end{aligned}$$

$$\nabla \int_0^t f(x + (s - t)b, s)\, ds = \int_0^t \nabla f(x + (s - t)b, s)\, ds.$$

The sum is equal to $f(x, t)$ as required. We find the initial value of this function by substituting $t = 0$. But then we have $\int_0^0$ which is always 0.

(d) Because this is a linear equation, the difference between two solutions of the inhomogeneous equation is a solution of the homogeneous equation with zero initial value. We have seen in lectures that the solution to the homogeneous transport equation with initial value is unique. Therefore any two solutions must be equal.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.