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Introduction to Partial Differential Equations
Exercise sheet 12

9th December, 2020

37. 1D Waves.

(a) Show that the a smooth function $u = u(\zeta, \eta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution to $\partial_{\zeta\eta} u = 0$ exactly when it is of the form $u(\zeta, \eta) = F(\zeta) + G(\eta)$, for smooth functions $F, G : \mathbb{R} \to \mathbb{R}$.
(2 Point(s))

(b) Under the parameterisation $\zeta = x + t, \eta = x - t$, show that u obeys the one dimensional wave equation $(\partial_{tt} - \partial_{xx})u = 0$ exactly when it solves the PDE in (a). *(2 Point(s))*

(c) From parts (a) and (b), derive D'Alembert's formula. *(2 Point(s))*

38. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_{tt} u - a^2 \partial_{xx} u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = \sin(x)$ and $h(x) = 1$. *(2+2 Point(s))*

39. Weak waves.

Let U be an open set in $\mathbb{R}^n$ and $\Omega = U \times (0, T)$ be a cylinder in $\mathbb{R}^{n+1}$. A continuous function u is called a *weak solution* of the wave equation on Ω if

$$\int_\Omega (\partial_{tt}\varphi - \Delta\varphi)\, u \, dx\, dt = 0$$

for every test function $\varphi \in C_0^\infty(\Omega)$. Solutions to the wave equation in the ordinary sense are called *classical* or *strong* in this context.

(a) Show that $u \in C^2(\Omega)$ is a weak solution if and only if it is a classical solution. *(3 Point(s))*

(b) Suppose that $(u_k)_{k\in\mathbb{N}}$ is a sequence of weak solutions that converges to u with local uniform continuity on Ω. Show that u is also a weak solution. *(4 Point(s))*

40. 1D Waves in the weak sense.

(a) Show that for given continuous functions F, G on $\mathbb{R}$, the function $u(x, t) = F(x+t) + G(x-t)$ is a weak solution of the one dimensional wave equation.
[Hint. Mollify F and G.] *(4 Point(s))*

(b) Show that the Fourier series

$$u(x, t) = \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)\, \sin kx,$$

where a_k and b_k are real sequences with $\sum |a_k| + |b_k| < \infty$, is a weak solution of the one dimensional wave equation. *(3 Point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.