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Introduction to Partial Differential Equations
Exercise sheet 10

2nd December, 2020

30. The heat kernel on $\mathbb{S}^1$.

(See also Exercise 4.22 in the lecture script) Denote the fundamental solution of the heat equation by $\Phi(x, t)$.

(a) Give the definition of a Schwartz function. *(1 Point(s))*

(b) Show that $f(x) = e^{-x^2}(e^{-x^2} + \sin^2 x)$ is a positive Schwartz function whose square root is not a Schwartz function. *(2 Points + 3 Bonus Points)*

(c) Show that for every $t > 0$ the function $\Phi(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is Schwartz function. *(3 Point(s))*

(d) Calculate the Fourier transform of $\Phi(\cdot, t)$ for any $t > 0$. [FYI. $\int_{\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$.] *(3 Point(s))*

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Show that

$$\tilde{f}(x) = \sum_{n \in \mathbb{Z}} f(x + n)$$

defines a smooth periodic function with period 1 (i.e. $\tilde{f}(x + 1) = \tilde{f}(x)$). *(2 Point(s))*

(f) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, with period 1, and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ a solution to the heat equation with initial condition $u(x, 0) = h(x)$. Show that u remains periodic in the spatial coordinate for all time. *(2 Point(s))*

(g) Conclude that

$$u(x, t) := \int_{\mathbb{S}^1} h(y) \sum_{n \in \mathbb{Z}} \Phi(x - y + n, t).$$

solves the heat equation with the initial condition. *(2 Point(s))*

(h) Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies

$$\sum_{n \in \mathbb{Z}} f(x + n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

Show, with the aid of this equality, the relation

$$\Theta(x - y, 4\pi i t) = \sum_{n \in \mathbb{Z}} \Phi(x - y + n, t),$$

where the left hand side is the Jacobi's theta function from Section 4.7.

(3 Bonus Point(s))

(i) How would you modify Definition 4.14 to give give an abstract definition of the heat kernel $H_{\mathbb{S}^1}$? *(3 Bonus Point(s))*

31. The connection between the fundamental solutions of the heat equation and the Laplace equation.

Let Φ_H be the fundamental solution of the heat equation and Φ_L the fundamental solution of the Laplace equation on $\mathbb{R}^n$ for $n \geq 3$. Denote the Laplace transformation of Φ_H by

$$G(x, \lambda) := \int_0^\infty \Phi_H(x,t) e^{-\lambda t}\, dt.$$

Show for any fixed $x \in \mathbb{R}^n \setminus \{0\}$ that $g(x) := \lim_{\lambda \to 0} G(x, \lambda)$ exists. Show that g has the form $a\Phi_L + b$ for constants $a, b \in \mathbb{R}$. *(5 Point(s))*

32. The heat kernel on $[0, 1]$**.**

(Exercise 4.23 from the lecture script)

(a) Show the final step in the calculation of the heat kernel $H_{[0,1]}$:

$$\sum_{k=1}^{\infty} e^{-\pi^2 k^2 t}(\cos(k\pi(x-y)) - \cos(k\pi(x+y))) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t)$$

(2 Points)

(b) Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z}+1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2. *(1 Point)*

(c) Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth functions $\tilde{f}$ in $\mathcal{A}$:

$$\tilde{f}(x) = \sum_{n \in \mathbb{Z}} f(2n+x) - \sum_{n \in \mathbb{Z}} f(2n-x).$$

(2 Points)

(d) Show for any $h \in \mathcal{A}$, that the solutions of the heat equation with initial value h is for all $t > 0$ a smooth function in $\mathcal{A}$. Conclude form this that the following sum has the properties of the Heat kernel of $[0, 1]$:

$$\sum_{n \in \mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n \in \mathbb{Z}} \Phi(x + 2n + y, t).$$

(3 Bonus Points)

(e) Show the relation

$$H_{[0,1]}(x, y, t) = \sum_{n \in \mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n \in \mathbb{Z}} \Phi(x + 2n + y, t),$$

where the left hand side the heat kernel in terms of theta functions as given in the lecture script. *(2 Bonus Points)*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.