Chapter 2

General Concepts

In this chapter we prepare for our investigation of the three main examples of linear second order partial differential equations in the subsequent three chapters.

2.1 Divergence Theorem

In this section we present the divergence theorem, which is a generalization of the fundamental theorem of calculus to higher dimensions. This will have many important consequences, but let us just mention two of them here: First we can generalize partial integration to higher dimensions. Second it allows us to understand the sense in which the higher dimensional scalar conservation law describes a conserved quantity. In order to state the theorem we have to describe how to integrate over submanifolds of $\mathbb{R}^n$. We start with a definition of such submanifolds.

Definition 2.1. *A subset $A \subset \mathbb{R}^n$ is called a k-dimensional submanifold if A is covered by the images O of homeomorphisms $\Phi : U \to O$ from open subsets $U \subset \mathbb{R}^k$ onto open subset of A, such that Φ considered as maps into $\mathbb{R}^n$ are continuously differentiable and have on U a Jacobian of full rank k.*

The Jacobian of Φ is a $n \times k$ matrix, whose rank cannot be greater than n, so $1 \leq k \leq n$. If $\Phi : U \to O$ has the properties in the definition, then choose for any $x \in O$ a k-dimensional linear subspace $V \subset \mathbb{R}^n$, such that the composition $P_V \circ \Phi'(\Phi^{-1}(x))$ with the orthogonal projection P_V onto V is bijective onto V. By the inverse function theorem we may diminish U and O such that $P_V \circ \Phi$ is a C^1-diffeomorphism from U onto an open subset W of V. The image O of Φ is the zero set of the function $W \times V^\perp \to V^\perp$, $(y, z) \mapsto z - (1 - P_V) \circ \Phi \circ (P_V \circ \Phi)^{-1}(y)$, which has rank $n - k$. We consider $W \times V^\perp \subset V \times V^\perp \simeq \mathbb{R}^n$ as an open subset of $\mathbb{R}^n$ which contains O. So the set A satisfies for any $x \in A$ the following condition: There exists on an open neighbourhood of x in $\mathbb{R}^n$ a continuously differentiable function to $\mathbb{R}^{n-k}$ whose Jacobian has at x rank $n-k$, such that the intersection of A with this neighbourhood is the level set of this function through x. Conversely, by the implicit function theorem, a subset $A \subset \mathbb{R}^n$ which satisfies the latter condition for all $x \in A$ is a k-dimensional submanifold. So we may alternatively characterize submanifolds by this latter condition.

The definition of an integral over submanifolds uses so called *partitions of unity.*

Definition 2.2. *(Partition of Unity) For a given family $(U_\alpha)_{\alpha\in A}$ of open subsets of $\mathbb{R}^n$ with union $\bigcup_{\alpha\in A} U_\alpha = \Omega \subset \mathbb{R}^n$ a smooth partition of unity is a countable family $(h_l)_{l\in\mathbb{N}}$ of smooth functions $h_l : \Omega \to [0,1]$ with the following properties:*

(i) *Each $x \in \Omega$ has a neighbourhood where all but finitely h_l vanish identically.*

(ii) *For all $x \in \Omega$ we have $\sum_{l=1}^{\infty} h_l(x) = 1$.*

(iii) *Each h_l vanishes outside a compact subset of U_α for some $\alpha \in A$.*

For every family of open subsets of $\mathbb{R}^n$ there exists a smooth partition of unity. A proof you can find in many textbooks and in my script of the lecture Analysis II.

Definition 2.3. *Let $A \subset \mathbb{R}^d$ be a k-dimensional submanifold of $\mathbb{R}^n$ and let $f \in C(A,\mathbb{R})$ vanish outside of a compact subset $K \subset A$. We cover K by finitely many open subsets $O \subset \mathbb{R}^d$ with $A \cap O = \Phi[U]$ for a map Φ as in Definition 2.1 and choose a corresponding partition of unity $(h_l)_{l\in\mathbb{N}}$. The integral of f over A is defined as*

$$\int_A f d\sigma = \sum\nolimits_{l\in\mathbb{N}} \int_U (h_l f) \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu.$$

Note that the volume of the k-dimensional parallelotope spanned by the column vectors of a $n \times k$ matrix A is equal to $\sqrt{\det(A^T A)}$. Here $A^T A$ is the matrix of all scalar products between the column vectors of A.

Lemma 2.4. *The integral $\int_A f d\sigma$ neither depends on the choice of the parametrizations $\Phi : U \to O$ in definition 2.1 nor on the choice of the partition of unity.*

Proof. Due to condition (i) on the partition of unity the sum in the definition of $\int_A f d\sigma$ is finite. For two covers of K by sets of the form $\Phi[U]$ and $\Psi[V]$ as in Definition 2.1 with corresponding partitions of unity, the intersections of two such sets (one from each cover) and the products of two functions (one from each partition of unity) build another cover of K with a corresponding partition of unity. The linearity of the integral and condition (ii) on the partition of unities together ensure that it suffices to consider the subcase that K is contained the images $\Phi[U]$ and $\Psi[V]$ of two continuously differentiable homeomorphisms as described in Definition 2.1. The restrictions of Φ to

$\Phi^{-1}[\Phi[U] \cap \Psi[V]]$ and of Ψ to $\Psi^{-1}[\Phi[U] \cap \Psi[V]]$ are both homeomorphisms onto the open subset $\Phi[U] \cap \Psi[V]$ of A. The composition of the second with the inverse of the first yields a homeomorphism $\Upsilon : \Psi^{-1}[\Phi[U] \cap \Psi[V]] \to \Phi^{-1}[\Phi[U] \cap \Psi[V]]$, such that $\Psi(x) = \Phi(\Upsilon(x))$ holds for all $x \in \Psi^{-1}[\Phi[U] \cap \Psi[V]]$.

Now we claim that Υ is continuously differentiable. For any $x \in \Psi^{-1}[\Phi[U] \cap \Psi[V]]$ there exists a k-dimensional linear subspace of $\mathbb{R}^n$ such that the composition $P \circ \Phi'(\Upsilon(x))$ with the orthogonal projection P onto this subspace is bijective onto this subspace. By the inverse function theorem an open neighbourhood of $\Upsilon(x)$ is mapped by $P \circ \Phi$ homeomorphically onto an open neighbourhood of $P(\Psi(x))$. The inverse mapping is together with $P \circ \Phi$ continuously differentiable. The map Υ is on this neighbourhood of x equal to the composition of $P \circ \Psi$ with the inverse map of $P \circ \Phi$, since $P \circ \Psi$ and $P \circ \Phi \circ \Upsilon$ coincide there. This shows that Υ is on this neighbourhood continuously differentiable. Because this is true for all $x \in \Psi^{-1}[\Phi[U] \cap \Psi[V]]$ the claim follows. We conclude

$$\int\limits_{\Psi^{-1}[\Phi[U]\cap\Psi[V]]} f \circ \Psi \sqrt{\det((\Psi')^T \Psi')} d\sigma = \int\limits_{\Psi^{-1}[\Phi[U]\cap\Psi[V]]} f \circ \Phi \circ \Upsilon \sqrt{\det(((\Phi \circ \Upsilon)')^T (\Phi \circ \Upsilon)'} d\sigma =$$

$$= \int\limits_{\Psi^{-1}[\Phi[U]\cap\Psi[V]]} \left(f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} \right) \circ \Upsilon |\det \Upsilon'| d\sigma = \int\limits_{\Phi^{-1}[\Phi[U]\cap\Psi[V]]} f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\sigma.$$

In the last step we applied the transformation formula of Jacobi. **q.e.d.**

In the divergence theorem we consider open subsets $\Omega \subset \mathbb{R}^n$ whose boundary are $n-1$-dimensional submanifolds. After the Definition 2.1 we explained how the implicit function theorem applies to these submanifolds. Since any $n-1$-dimensional subspace is the immage of $\mathbb{R}^{n-1} \simeq \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ with respect to some linear rotation O of $\mathbb{R}^n$ these arguments show that the homoemorphisms in Definiton 2.1 are of the form

$$\Phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n, \quad y \mapsto \mathsf{O}(x, g(x)) \text{ for some } C^1\text{-function } g : U \to (a, b) \subset \mathbb{R},$$
$$\text{with } \det((\Phi'(x))^T \Phi'(x)) = \det \left(\begin{pmatrix} \mathbb{1} & \nabla g(x) \end{pmatrix} \mathsf{O}^T \mathsf{O} \begin{pmatrix} \mathbb{1} \\ \nabla^T g(x) \end{pmatrix} \right) = 1 + (\nabla g(x))^2.$$

The plane tangent to $\partial\Omega$ in $\mathsf{O}(x, g(x))$ is image of the kernel of the derivative of $U \times (a, b) \to \mathbb{R}$, $(x, z) \mapsto z - g(x)$ with respect to O. So

$$N(x, g(x)) = \frac{\mathsf{O}^T(-\nabla^T g(x), 1)}{\sqrt{1 + (\nabla g(x))^2}}$$

is up to sign unique normalized vector orthogonal the the tangent plane which is called normal. Since the last component is positive this normal points outwards of $\mathsf{O}[\{(y, z) \in U \times (a, b) \mid z < g(y)\}]$, which for an appropriate O is equal to $\Omega \cap \mathsf{O}[U \times (a, b)]$.

Theorem 2.5. *(Divergence Theorem) Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Let $f : \bar{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇f continuously extends to $\partial\Omega$. Then we have*

$$\int_\Omega \nabla \cdot f \,\mathrm{d}\mu = \int_{\partial\Omega} f \cdot N \,\mathrm{d}\sigma$$

Here N is the outward-pointing normal and $N\,\mathrm{d}\sigma$ the corresponding measure on $\partial\Omega$.

Proof. We cover $\bar{\Omega}$ by open subsets $U \times (a,b) \subset \mathbb{R}^n$ as described above and Ω. We choose an compatible partition of unity. Due to the compactness of $\bar{\Omega}$ and due to condition (iii) on the partition of unity this partition has only finitely many members. By linearity it suffices to show the statement for any term individually.

First we consider a continuously differentiable function $f : \Omega \to \mathbb{R}^n$ with compact support in Ω. By setting it zero outside of Ω it extends continuously differentiable to $\mathbb{R}^n$. Choose a cartesian product of finite intervalls which contains Ω. The continued function vanishes on the boundary of this box. By Fubini we may integrate the i-th term of $\nabla \cdot f = \partial_1 f_1 + \ldots, +\partial_n f_n$ first dx_i. Due to the fundamental theorem of calculus this integral is the difference of the values of f at two boundary points and vanishes. This shows that in this case both sides of the divergence theorem vanish.

Now we consider a function f on $\Omega \cap O[U \times (a,b)] = \{O(x,z) \mid z \leq g(x)\}$ which vanishes outside a compact subset. We replace x by Ox, $x \mapsto f(x)$ by $x \mapsto O^T f(Ox)$, $x \mapsto N(x)$ by $x \mapsto O^T N(Ox)$ and $\Omega \ni Ox \Leftrightarrow O^{-1}[\Omega] \ni x$. Consequently $O^T O = \mathbb{1}$, $\det O = \pm 1$ and $\nabla \cdot O^T f(Ox) = \mathrm{trace}(O^T \circ f \circ O)'(x) = \mathrm{trace}(OO^T f'(Ox) = \nabla \cdot f(Ox)$. By Jacobi's transformation formula both sides of the divergence theorem do not change, and we may omit O. Again we extend f to $\mathbb{R}^{d-1} \times (a,b)$ by setting it zero outside of $U \times (a,b)$. For any $(x,y) \in \mathbb{R}^{d-1} \times (a,b)$, $1 \leq i < n$ we have

$$\int_a^y \int_{-\infty}^0 \frac{\partial}{\partial x_i} f(x + te_i, z) dt dz = \int_a^y f(x,z) dz$$

By Fubini this function is continuously differentiable with

$$\frac{\partial}{\partial x_i} \int_a^y f(x,z) dz = \int_a^y \frac{\partial f(x,z)}{\partial x_i} dz \text{ für } 1 \leq i < d, \quad \frac{\partial}{\partial y} \int_a^y f(x,z) dz = f(x,y).$$

The following function vanishes outside a compact subset of U:

$$x \mapsto \int_a^{g(x)} f(x,z) dz \quad \text{mit} \quad \frac{\partial}{\partial x_i} \int_a^{g(x)} f(x,z) dz = \int_a^{g(x)} \frac{\partial f(x,z)}{\partial x_i} dz + \frac{\partial g(x)}{\partial x_i} f(x, g(x)).$$

So the arguments of the first case apply and show that the integral over U on the right hand side vanishes. This proves for $1 \leq i < n$ the divergence theorem:

$$\int_U \int_a^{g(x)} \frac{\partial f_i(x,z)}{\partial x_i} dz d^{d-1}x = -\int_U f_i(x,g(x)) \frac{\partial g(x)}{\partial x_i} d^{d-1}x = \int_U f_i(x,g(x)) N_i(x,g(x)) d\sigma.$$

The fundamental theorem of calculus finishes the proof, since f vanishes on $U \times \{a\}$:

$$\int_U \int_a^{g(x)} \frac{\partial f_n(x,z)}{\partial x_n} dz d^{d-1}x = \int_U f_n(x,g(x)) d^{d-1}x = \int_U f_n(x,g(x)) N_n(x,g(x)) d\sigma. \text{ \textbf{q.e.d.}}$$

The divergence theorem implies for all $i = 1, \ldots, n$

$$\int_\Omega \partial_i f \, \mathrm{d}\mu = \int_{\partial\Omega} f N_i \, \mathrm{d}\sigma$$

For two functions f and g whose product vanishes on the boundary $\partial\Omega$ and satisfies the corresponding assumptions of the divergence theorem we obtain by the Leibniz rule

$$\int_\Omega f \partial_i g \, \mathrm{d}\mu = -\int_\Omega g \partial_i f \, \mathrm{d}\sigma \qquad \text{for all } i = 1, \ldots, n.$$

This is called integration by parts. Inductively we get for any multiindex γ

$$\int_\Omega f \partial^\gamma g \, \mathrm{d}\mu = (-1)^{|\gamma|} \int_\Omega g \partial^\gamma f \, \mathrm{d}\sigma.$$

As a second application of the divergence theorem we present conserved quantities for any continuously differentiable function $f : \mathbb{R} \to \mathbb{R}^n$ and any solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the general scalar conservation law introduced in the last chapter

$$\dot{u}(x,t) + \nabla f(u(x,t)) = \dot{u}(x,t) + f'(u(x,t)) \cdot \nabla u(x,t) = 0.$$

For open $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ being a $n-1$-dimensional submanifolds of $\mathbb{R}^n$ we obtain

$$\frac{d}{dt} \int_\Omega u(x,t) d^n x = \int_\Omega \dot{u}(x,t) d^n x = -\int_\Omega \nabla f(u(x,t)) d^n x = -\int_{\partial\Omega} f(u(x,t)) \cdot N(x) d\sigma(x).$$

This is the meaning of a conservation law: the change of the integral of $u(\cdot,t)$ over $\Omega \subset \mathbb{R}^n$ is equal to the integral of the flux $-f(u(\cdot,t)) \cdot N$ through the boundary $\partial\Omega$.

2.2 Classification of Second order PDEs

For PDEs of order greater then one, there does not exists a general theory. We shall present in Section 2.3 an example of a PDE with smooth coefficients, which has in a neighbourhood of some point no solutions at all. Over the time there have been discovered different methods to solve several PDEs, in particular those PDEs which show up in physics. Afterwards these methods were extended to larger and larger classes of PDEs. It turned out that the successful methods of solving PDEs differ from each other substantially. As a result there does not exists one unified theory of PDEs, but there exist several islands of well understood families of PDEs inside the large set of all PDEs. It was Jacobi who formulated in his lectures on Dynamics in the years 1842-43 the following general recipe:

"The main obstacle for the integration of a given differential equations lies in the definition of adapted variables, for which there is no general rule. For this reason we should reverse the direction of our investigation and should endeavour to find, for a successful substitution, other problems which might be solved by the same."

The strategy is to determine for any successful method all PDEs which can be solved by this method. We already presented for the first order PDEs a more or less general method. Now we investigate the second order PDEs. In this lecture we consider only second order linear PDEs. A general second order linear PDE has the following form

$$Lu(x) = \sum\nolimits_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u + \sum\nolimits_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x) = 0.$$

By Schwarz's Theorem for twice differentiable u this expression does not change if we replace a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$. So we may assume that a_{ij} is symmetric and diagonalizable.

Elliptic PDEs. If the matrix a_{ij} is the unity matrix and $b = 0 = c$, then this is the

Laplace equation. $$\triangle u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Solutions of the Laplace equation are called harmonic functions. In Chapter 3 we present several tools which establish many properties of these harmonic functions. It turns out that many properties of the harmonic functions also apply to general solutions of $Lu = 0$, if the matrix a_{ij} is positive (or negative) definite. These are the main examples of the so called elliptic PDEs. There has been done a lot of work to extend these tools to larger and larger classes of elliptic PDEs. One of the results is that the influence of the higher order derivatives on the properties of solutions is much more important than the influence of the lower order derivatives. An important tool are so called a priori estimates. Such estimates show that the lower order derivatives can be estimated in terms of the second order derivatives. We offer another lecture which presents many of these tools for such elliptic second order PDEs.

Beside the linear elliptic PDEs there are also non-linear PDEs, to which these methods of elliptic PDEs apply. An important example whose investigation played a major role in the development of the elliptic theory is the

Minimal surface equaltion. $\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0, \quad u : \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^n$ open.

The graphs of solutions describe so called minimal surfaces. The area of such hypersurfaces in $\mathbb{R}^{n+1}$ does not change with respect to infinitesimal variations. Soap bubbles are examples of such minimal surfaces. The boundary value problem of the minimal surface equation is called Plateau's problem. For the first proof of the existence of solutions of this Plateau problem in the 1930s, Jesse Douglas received the first Field's Medal. In this non-linear second order PDE the coefficients of the second derivatives also depend on the solution. A lot of work has been done to extend the tools of elliptic theory to elliptic PDEs whose coefficients belong to larger and larger functions spaces. This development induced the introduction of many new function spaces. In Section 2.4 we shall introduce the so called space of distributions. Many of the more advanced functions spaces are build on the base of these spaces.

Parabolic PDEs. For these linear PDEs the matrix a_{ij} considered as a symmetric bilinear form is only semi-definite and they belong to the boundary of the class of elliptic PDEs. Most of the methods of elliptic PDEs have an extension to this limiting case. So these limiting cases together with the class of elliptic PDEs form some extended class of elliptic PDEs. Of particular importance is the subclass of linear PDEs with semi-definite matrices a_{ij} which have a one-dimensional kernel. Since symmetric matrices are always diagonalizable this means that one eigenvalue of a_{ij} vanishes and all other eigenvalues have the same sign. In spite of the deep relationship to the elliptic PDEs these equations have their own label: parabolic PDEs. The simplest example is the

Heat equation. $\dot{u} - \triangle u = 0.$

These parabolic PDEs describe diffusion processes. These are processes which level inhomogeneities of some quantity by some flow along the negative gradient of the quantity. A typical example for this quantity is the temperature form which the name for the heat equation originates. Many stochastic processes have this property. So the theory of parabolic PDEs has a deep relationship to the theory of stochastic processes. In this lecture we present in Chapter 4 this simplest example of linear parabolic PDE. We shall see how the tools for the Laplace equation can be applied in modified form to this heat equation. In case of the parabolic PDEs there too exists a non-linear example from the geometric analysis, whose investigation played a major role for the development of the elliptic theory (the tensor fields g and R are defined below):

Ricci Flow. $\dot{g}_{ij} = -2R_{ij}.$

This PDE describes a diffusion-like process on Riemannian manifolds. It levels the inhomogeneities of the metric, namely the Riemannian metric g. In the long run the corresponding Riemannian manifolds converge to metric spaces with large symmetry groups. Richard Hamilton proposed (in the 1970s) a program that aims to prove the geometrization conjecture of Thurston with the help of these PDEs. It states that every three-dimensional manifold can be split into parts, which can be endowed with an Riemannian metric such that the isometry group acts transitively. This conjecture implies the Poincare conjecture, which states that every simply connected compact manifold is the 3-sphere. Hamilton tries to control the long time limit of the Ricci flow on a general 3-dimensional Riemannian manifold. In 2003 the Russian mathematician Grisha Perelman published on the internet three articles which overcome the last obstacle of this program. This lead to the first proof of one of the Millennium Problems of the American Mathematical Society and was a great success of geometric analysis.

Hyperbolic PDEs. Besides the elliptic PDEs (including the limiting cases) the second important class of linear PDEs are called hyperbolic. In this case the matrix a_{ij} has one eigenvalue of opposite sign than all other eigenvalues. The simplest example is the

Wave equation. $$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0.$$

In Chapter 5 we present the methods how to solve this equation. We shall see that it describes the propagation of waves with constant finite speed. The solutions of general hyperbolic equations are similar to the solutions of this case, and many tools can be generalized to all hyperbolic PDEs. The investigation of these PDEs depend on the understanding of all trajectories, which propagate by the given speed. It was motivated by theory of the electrodynamic fields, whose main system of PDEs are the

Maxwell equations. $$\begin{aligned} \dot{E} - \nabla \times B &= -4\pi j & \dot{B} + \nabla \times E &= 0 \\ \nabla \cdot E &= 4\pi\rho & \nabla \cdot B &= 0. \end{aligned}$$

In this theory there is given a distribution of charges ρ and currents j on space time $\mathbb{R}\times\mathbb{R}^3$. The unknown functions are the electric magnetic fields E and B, which describe the electrodynamic forces induced by the given distributions of charges and currents ρ and j. The conservation of charge is formulated in the same way as in the scalar conservation law. So the change of the total charge contained in a spatial domain is described by the flux of the current through the boundary of the domain. By the divergence theorem this means that distributions of charge ρ and currents j obey

$$\dot{\rho} + \nabla \cdot j = 0.$$

Again there exists a non-linear version which stimulated the development of the theory:

Einsteins field equations of general realitivity. $$R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij}.$$

Here for a given distribution of masses the energy stress tensor and the space time metric g_{ij} are the unknown functions. This metric is a symmetric bilinear form with one positive and three negative eigenvalues on the tangent space of space time. The corresponding Ricci curvature is denoted by R_{ij} and the scalar curvature by R:

$$\Gamma^k_{ij} := \frac{1}{2}\sum\nolimits_{l=0}^{3} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right), \quad \left(g^{ij}\right) := (g_{ij})^{-1} \text{ inverse metric}$$

$$R_{ij} := \sum\nolimits_{k=0}^{3} g^{kl}\left(\frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \sum\nolimits_{l=0}^{3}\left(\Gamma^k_{lk}\Gamma^l_{ij} - \Gamma^k_{lj}\Gamma^l_{ik}\right)\right), \quad R := \sum\nolimits_{i,j=0}^{3} g^{ij}R_{ij}.$$

Integrable Systems with Lax operators. Finally I want to mention a smaller class of PDEs, which are the main objects of my research. They are non-linear PDEs which describe an evolution with respect to time which is very stable. This means that the solutions have in a specific sense a maximal number of conserved quantities. The theory of integrable systems belongs to the field of Hamiltonian mechanics, which originated from Newtons description of the motion of the planets. The Scottish Lord John Scott Russell got very excited in 1934 about the observation of an solitary wave in a Scottish channel and published a "Report on Waves". This report was quite influential. The two Dutch mathematicians Korteweg and De Vries translated his observation into a PDE describing the profile of water waves travelling along the channel:

Korteweg-de-Vries equation. $$4\dot{u} - 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0.$$

First by numerical experiments in the 1950s with the first computers and latter in the 1970s by mathematical theory, the solutions of this PDE were shown to have exactly the properties which made Lord Russell so exited: they described waves which propagate through each other without changing their shape. This lead to the discovery of an hidden relation of the theory of integrable systems with the theory of Riemann surfaces, which is another field with a long history. A major step towards the discovery of this relation was the observation of Peter Lax that this equation can be written as

$$\dot{L} = [A, L] \qquad \text{with} \qquad L := \frac{\partial^2}{\partial x^2} + u \qquad A := \frac{\partial^3}{\partial x^3} + \frac{3u}{2}\frac{\partial}{\partial x} + \frac{3}{4}\frac{\partial u}{\partial x}.$$

2.3 Existence of Solutions

In order to demonstrate the difference between ODEs and PDEs we shall present an example of a partial differential equation with smooth coefficients without solutions. This example is a simplification by Nirenberg of an example of H. Levy.

For a given complex-valued function f on a domain $(x, y) \in \mathbb{R}^2$ we look for a complex valued solution u on the same domain of the following differential equations:

$$\frac{\partial u}{\partial x} + \imath x\frac{\partial u}{\partial y} = f(x, y).$$

We impose the following two conditions on the smooth function f:

(i) $f(-x, y) = f(x, y)$

(ii) there exists a sequence of positive numbers $\varrho_n \downarrow 0$ converging to zero, such that f vanishes on a neighbourhood of the circles $\partial B(0, \varrho_n)$ in contrast to non-vanishing integrals $\int_{B(0,\varrho_n)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \neq 0$.

If $h : \mathbb{R} \to [0, \infty)$ is a smooth periodic function vanishing on an intervall but not on $\mathbb{R}$, then $f(x) := \exp(-1/|x|)h(1/|x|)$ has these two properties.

Now we shall prove by contradiction that there exists no continuously differentiable solution u in a neighbourhood of $(0, 0) \in \mathbb{R}^2$.

Step 1: If the function $u(x, y)$ is a solution, then due to (i) $-u(-x, y)$ is also a solution. Hence we may replace $u(x, y)$ by $\frac{1}{2}(u(x, y) - u(-x, y))$ and assume $u(-x, y) = -u(x, y)$.

Step 2: We claim that every solution u vanishes on the circles $\partial B(0, \varrho_n)$. In fact, we transform small annuli A onto domains $\tilde{A}$ in $\mathbb{R}^2$:

$$A \to \tilde{A}, \qquad (x, y) \mapsto \begin{cases} (x^2/2, y) & \text{for } x \geq 0 \\ (-x^2/2, y) & \text{for } x < 0. \end{cases}$$

These transformations are homeomorphisms from A onto $\tilde{A}$. On the sub domains $\tilde{A}_+ = \left\{(s, y) \in \tilde{A} \mid s > 0\right\}$ the function $\tilde{u}(s, y) = u(x^2/2, y)$ is holomorphic:

$$2\bar{\partial}\tilde{u} = \frac{\partial \tilde{u}(s, y)}{\partial s} + \imath \frac{\tilde{u}(s, y)}{\partial y} = \frac{dx}{ds}\frac{\partial u(x, y)}{\partial x} + \imath \frac{\partial u(x, y)}{\partial y} = \frac{1}{x}\left(\frac{\partial u(x, y)}{\partial x} + \imath x \frac{\partial u(x, y)}{\partial y}\right) = 0.$$

Due to step 1. the function $\tilde{u}$ vanises on the line $s = 0$. This implies that $\tilde{u}$ together with the Taylor series vanishes identically on $\tilde{A}_+$ and due to step 1 on $\tilde{A}$.

Step 3: The Divergence Theorem yields a contradiction to the assumption (ii):

$$\begin{aligned} \int_{B(0,\varrho_n)} f\, \mathrm{d}x\, \mathrm{d}y &= \int_{B(0,\varrho_n)} \left(\frac{\partial u}{\partial x} + \imath x \frac{\partial u}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = \int_{B(0,\varrho_n)} \nabla \cdot \begin{pmatrix} u \\ \imath x u \end{pmatrix} \mathrm{d}x\, \mathrm{d}y \\ &= \int_{\partial B(0,\varrho_n)} \begin{pmatrix} u \\ \imath x u \end{pmatrix} \cdot N(x, y)\, \mathrm{d}\sigma(x, y) = 0, \end{aligned}$$

Therefore the given differential equation has no continuously differentiable solution.

This example also implies that the following partial differential equation with smooth real coefficients has no four times differentiable real solution:

$$\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \imath x \frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\tilde{u} = \left(\left(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right)\tilde{u} = f.$$

Here f is a real smooth function with the properties (i) and (ii). For any real solution $\tilde{u}$, the following complex function would be a solution of the complex PDE:

$$u = \left(\frac{\partial}{\partial x} - \imath x \frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\tilde{u}.$$

2.4 Distributions

Our investigation of partial differential equations aims to find as many solutions as possible and, in addition, conditions which uniquely determines the solutions. The existence and uniqueness of solutions depends on the notion of solution we use. Clearly all partial derivatives of a solution which occur in the partial differential equation have to exist. We might use several possible generalisations of derivatives in order to define such solutions. In this section we introduce generalised functions which can always be differentiated infinitely many times. For this achievement we have to pay a price: these generalised functions cannot be multiplied with each other. Linear partial differential equations extend to well defined equations on such generalised functions. Generalised functions solving the linear partial differential equations are called *weak solutions* or solutions in the sense of distributions. There exist other notions of weak solutions which also apply to non-linear partial differential equations. An example of more general functions with finitely many derivatives are so called Sobolev spaces. These Sobolev spaces are introduced in more advanced lectures on partial differential equations. The elements of the Sobolev spaces are distributions. So the distributions which we introduce now are the most general functions with derivatives.

The support $\operatorname{supp} f$ of a function f is the closure of $\{x \mid f(x) \neq 0\}$. On an open set $\Omega \subseteq \mathbb{R}^n$ let $C_0^\infty(\Omega)$ denote the algebra of smooth functions whose support is a compact subset of Ω. We say such functions have compact support in Ω and we use the notation $\operatorname{supp} f \Subset \Omega$. Each $f \in L^1(\Omega)$ defines a linear map

$$F : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \int_\Omega f\phi \, \mathrm{d}\mu.$$

Generalised funtions on Ω are such linear forms F on $C_0^\infty(\Omega)$. When considering the elements of $C_0^\infty(\Omega)$ as the domain of the linear form F, we call them test functions. If f has a derivative, then by integration by parts we obtain

$$\int_\Omega \partial_i f \phi \, \mathrm{d^n}x = -\int_\Omega f \partial_i \phi \, \mathrm{d^n}x.$$

For any linear form F on $C_0^\infty(\Omega)$ we define the partial derivatives as

$$\partial_i F : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto -F(\partial_i \phi).$$

Therefore such generalised functions have infinitley many derivatives. The vector space of test functions is infinite dimensional. In order to avoid abstract nonsense we should impose some continuity on the linear forms F. The derivative of a continuous functional F is again continuous, if the derivatives are linear continuous maps on the space $C_0^\infty(\Omega)$.

For $f \in L^1(\Omega))$ the corresponding linear functionals F are continuous with respect to the supremum norm on compact subsets of Ω. We define for any compact subset $K \subset \Omega$ and every multiindex α the following semi-norm:

$$\|\cdot\|_{K,\alpha} : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \|\phi\|_{K,\alpha} := \sup_{x \in K} |\partial^\alpha \phi(x)| \,.$$

Definition 2.6. *On an open subset $\Omega \subseteq \mathbb{R}^n$ the space of distributions $\mathcal{D}'(\Omega)$ is defined as the vector space space of all linear maps $F : C_0^\infty(\Omega) \to \mathbb{R}$ which are continuous with respect to the semi norms $\|\cdot\|_{K,\alpha}$; i.e. for each compact $K \subset \Omega$ there exist finitley many multi indices $\alpha_1, \ldots, \alpha_M$ and constants $C_1 > 0, \ldots, C_M > 0$ such that the following inequality holds for all test functions $\phi \in C_0^\infty(\Omega)$ with compact support in K:*

$$|F(\phi)| \leq C_1 \|\phi\|_{K,\alpha_1} + \ldots + C_M \|\phi\|_{K,\alpha_M}.$$

The support $\operatorname{supp} F$ of a distribution $F \in \mathcal{D}'(\Omega)$ is defined as the complement of the union of all open subsets $O \subset \Omega$, such that F vanishes on all test functions ϕ whose support is contained in O. We denote the Euclidean length of $x \in \mathbb{R}^n$ by $|x|$. The test function

$$\phi(x) := \begin{cases} \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

has support $\overline{B(0,1)}$ and is non-negative. By rescaling of x and ϕ and by translations we obtain for each ball $B(x_0,\epsilon)$ a non-negative test function $\phi_{B(x_0,\epsilon)}$ with $\operatorname{supp} \phi_{B(x_0,\epsilon)} = \overline{B(x_0,\epsilon)}$ with $\int \phi_{B(x_0,\epsilon)} \, \mathrm{d}\mu = 1$. In particular, there exists for every open subset $O \subset \Omega$ a non-negative test function with support contained O. Every continuous function f on Ω which does not vanish identically takes values in $(-\infty, \epsilon)$ or (ϵ, ∞) for some $\epsilon > 0$ on some properly chosen open ball. Therefore there exists $\phi \in C_0^\infty(\Omega)$ with $\int_\Omega f\phi \, \mathrm{d}\mu \neq 0$.

The following distribution does not correspond to a usual function:

$$\delta : C_0^\infty(\Omega) \to \mathbb{R} \qquad \phi \mapsto \phi(0).$$

A corresponding function would vanish on $\mathbb{R}^n \setminus \{0\}$ and would have a total integral one. Since $\{0\}$ has maesure zero such a function does not exist. This generalised function is called Dirac's δ-function. We shall see that the familiy of distributions which corresponds to the functions $\phi_{B(0,\epsilon)}$ converge in the limit $\epsilon \downarrow 0$ to this dirtribution. The support of all derivatives of this distribution contains only the point $0 \in \Omega$.

The product of a distribution with a function $g \in C^\infty(\Omega)$ is defined as

$$gF : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(g\phi).$$

This product makes the embedding $C^\infty(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ to a homomorphism of modules over the algebra $C^\infty(\Omega)$. However, even the product of a distribution with a continuous non-smooth functions is not defined. The convolution is another product on $C_0^\infty(\mathbb{R}^n)$:

$$(g * f)(x) := \int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^n y = \int_{\mathbb{R}^n} g(y)f(x-y)\,\mathrm{d}^n y.$$

This product is commutative and associative (Exercise). In order to extend this product to a product between a smooth function and a distribution we calculate:

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \phi(g*f)\,\mathrm{d}^n x &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \phi(x)g(x-y)f(y)\,\mathrm{d}^n y\,\mathrm{d}^n x = \int\limits_{\mathbb{R}^n}\phi(x)\int\limits_{\mathbb{R}^n}(\mathsf{T}_x\mathsf{P}g)(y)f(y)\,\mathrm{d}^n y\,\mathrm{d}^n x\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(x)g(x-y)f(y)\,\mathrm{d}^n x\,\mathrm{d}^n y = \int_{\mathbb{R}^n}(\phi*\mathsf{P}g)f\,\mathrm{d}^n y\end{aligned}$$

$$\text{with } \mathsf{T}_x : C_0^\infty(\Omega) \to C_0^\infty(x+\Omega), \quad \phi \mapsto \mathsf{T}_x\phi, \text{ and } (\mathsf{T}_x\phi)(y) = \phi(y-x)$$
$$\text{and } \mathsf{P} : C_0^\infty(\Omega) \to C_0^\infty(-\Omega), \quad \phi \mapsto \mathsf{P}\phi, \text{ with } (\mathsf{P}\phi)(y) = \phi(-y).$$

Therefore we define for $g \in C_0^\infty(\mathbb{R}^n)$ and $F \in \mathcal{D}'(\mathbb{R}^n)$

$$g*F : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto F(\mathsf{T}_x\mathsf{P}g) \text{ or equivalently } g*F : C_0^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad \phi \mapsto F(\phi*\mathsf{P}g).$$

Lemma 2.7. *The convolutions of a distribution F with a test function $g \in C_0^\infty(\Omega)$ is a distribution which corresponds to a smooth function. The support of this distribution is contained in the point-wise sum of the supports of the functions and the distribution.*

Proof. For each $F \in \mathcal{D}'(\Omega)$ the linearity and continuity imply

$$g*F(\phi) = F(\mathsf{P}g*\phi) = \int_{\mathbb{R}^n} F(\mathsf{T}_x\mathsf{P}g)\phi(x)\,\mathrm{d}^n x.$$

Due to the continuity of F with respect to the semi norms $\|\cdot\|_{K,0}$ the functions $x \mapsto F(\mathsf{T}_x\mathsf{P}g)$ are continuous. Furthermore, these functions are smooth since $\frac{\mathsf{T}_{y+\epsilon h}-\mathsf{T}_y}{\epsilon}\phi = \mathsf{T}_y\frac{\mathsf{T}_{\epsilon h}-\mathbb{1}}{\epsilon}\phi$ converges for all $\phi \in C^\infty(\mathbb{R}^n)$ in the limit $\epsilon \to 0$ on the space $C^\infty(\mathbb{R}^n)$ with respect to the topology induced by the semi norms $\|\cdot\|_{K,\alpha}$ to $\mathsf{T}_y\left(\sum_{i=1}^n h_i\partial_i\phi\right)$.

If $x \mapsto F(T_xPg)$ does not vanish on a neighbourhood of a point x, then $g(x-y) \neq 0$ for an element $y \in \operatorname{supp} F$. Hence $x = y + (x-y)$ is the sum of an element of $\operatorname{supp} F$ with an element of $\operatorname{supp} g$. **q.e.d.**

This Lemma implies that even the convolution of a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ with a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ with compact support $\operatorname{supp} G$ is a well defined distribution:

$$F*G : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(\phi*\mathsf{P}G) \text{ with } \qquad \mathsf{P}G(\phi) := G(\mathsf{P}\phi).$$

In particular, the δ-distribution is the neutral element of the product defined by the convolution, i.e. the convolution with the δ-distribution maps each distribution to itself. We introduced a family of test functions $(\phi_{B(0,\epsilon)})_{\epsilon>0}$ which converge in the limit $\epsilon \downarrow 0$ to the δ-distribution. For each $F \in \mathcal{D}'(\Omega)$ the family $f_\epsilon := \phi_{B(0,\epsilon)} * F$ converge in the limit $\epsilon \downarrow 0$ in a specific sense to F. Such a family $(\lambda_\epsilon)_{\epsilon>0}$ in $C_0(\mathbb{R}^n)$ with

$$\lambda_\epsilon \geq 0 \qquad\qquad \operatorname{supp}\lambda_\epsilon \subset \overline{B(0,\epsilon)} \qquad\qquad \int_{\mathbb{R}^n} \lambda_\epsilon \, \mathrm{d}^{\mathrm{n}}x = 1,$$

which converges in the limit $\epsilon \downarrow 0$ to the δ-distribution, is called mollifier. Now we can show that all distributions can be approximated by smooth functions.

Lemma 2.8. *Let $f \in C(\Omega)$ and $(\lambda_\epsilon)_{\epsilon>0}$ be a mollifier. In the limit $\epsilon \downarrow 0$ the family of smooth functions $\lambda_\epsilon * f$ converges uniformly on compact subets of Ω to f. For smooth functions the same holds for all derivatives of f.*

Proof. On compact sets continuous functions are uniformly continuous. Any $x \in \Omega$ is contained in an open ball $B(x,\epsilon) \subset \Omega$. For sufficiently small ϵ we have

$$|(\lambda_\epsilon * f)(x) - f(x)| = \left| \int_{B(x,\epsilon)} \lambda_\epsilon(x-y)(f(y)-f(x)) \, \mathrm{d}^{\mathrm{n}}y \right| \leq \sup_{y \in B(x,\epsilon)} |f(y) - f(x)|.$$

This shows the uniform convergency $\lim_{\epsilon\downarrow 0} \lambda_\epsilon * f = f$. By definition of the convolution two smooth functions f and g obey

$$\partial_i(f * g) = f * \partial_i g = \partial_i f * g.$$

Hence for $f \in C^\infty(\Omega)$ these arguments carry over to all partial derivatives of f. **q.e.d.**

As previously mentioned, any $f \in L^1_{\mathrm{loc}}(\Omega)$ defines in a canonical way a distribution

$$F_f : C_0^\infty(\Omega) \to \mathbb{R}, \qquad\qquad \phi \mapsto \int_\Omega f\phi \, \mathrm{d}\mu.$$

For $\phi \in C_0^\infty(\Omega)$ with support in a compact subset $K \subset \Omega$ and $f \in L^1(\Omega)$ gwe have

$$|F_f(\phi)| \leq \sup_{x \in K} |\phi(x)| \|f\|_{L^1(\Omega)}.$$

For $f \in L^1_{\mathrm{loc}}(\Omega)$ every compact subset $K \subset \Omega$ has a cover of open subsets $O_1, \ldots, O_L$ of Ω such that $f|_{O_l} \in L^1(O_l)$ for $l = 1, \ldots, L$. This shows $F_f \in \mathcal{D}'(\Omega)$:

$$|F_f(\phi)| \leq \sup_{x \in K} |\phi(x)| \sum_{l=1}^{L} \|f|_{O_l}\|_{L^1(O_l)} \quad \text{for} \quad \operatorname{supp}\phi \subset K.$$

Lemma 2.9. *(Fundamental Lemma of the Calculus of Variantions) If $f \in L^1_{\mathrm{loc}}(\Omega)$ obeys $F_f(\phi) \geq 0$ for all non-negative test functions $\phi \in C_0^\infty(\Omega)$, then f is non-negative almost everywhere. In particular the map $L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto F_f$ is injective.*

Proof. It suffices to prove the local statement for $f \in L^1(\Omega)$. We extend f to $\mathbb{R}^n$ by setting f on $\mathbb{R}^n \setminus \Omega$ equal to zero. The extended function is also denoted by f and belongs to $f \in L^1(\mathbb{R}^n)$. For a mollifier $(\lambda_\epsilon)_{\epsilon>0}$ we have

$$\begin{aligned}\|\lambda_\epsilon * f - f\|_1 &= \int_{\mathbb{R}^n} \left| \int_{B(0,\epsilon)} \lambda_\epsilon(y) f(x-y)\,\mathrm{d}^{\mathrm{n}}y - f(x) \right| \mathrm{d}^{\mathrm{n}}x \leq \\ &\leq \int_{B(0,\epsilon)} \int_{\mathbb{R}^n} \lambda_\epsilon(y)|f(x-y) - f(x)|\,\mathrm{d}^{\mathrm{n}}x\,\mathrm{d}^{\mathrm{n}}y \leq \sup_{y \in B(0,\epsilon)} \|f(\cdot - y) - f\|_1.\end{aligned}$$

If f is the chracteristic functions of a rectangle, then the supremum on the right hand side converges to zero for $\epsilon \downarrow 0$. Due to the triangle inequality the same holds for step functions, i.e. finite linear combinations of such functions. Since step functions are dense in $L^1(\mathbb{R}^n)$ for each $f \in L^1(\mathbb{R}^n)$ this supremum becomes abitrary small for sufficiently small ϵ. Hence the family of functions $(\lambda_\epsilon * f)_{\epsilon>0}$ converges in $L^1(\mathbb{R}^n)$ in the limit $\epsilon \downarrow 0$ to f. Hence there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ which converges to zero, with $\|f_{n+1} - f_n\|_1 \leq 2^{-n}$ for all $n \in \mathbb{N}$ and $f_n = \lambda_{\epsilon_n} * f$. This ensures that the series $|f_1| + \sum_{n\in\mathbb{N}} |f_{n+1} - f_n|$ converges in $L^1(\mathbb{R}^n)$. Furthermore, due to Lebesgue's bounded convergence theorem the sequence $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to f. The non-negativity of the mollifiers together with the assumption on F_f implies $(\lambda_\epsilon * f)(x) = F_f(\lambda_\epsilon(x - \cdot) \geq 0$. This indeed shows that f is a.e. non-negative.

In particular, if f belongs to the kernel of $f \mapsto F_f$, then f is almost everywhere non-negative and non-positive. So f vanishes almost everywhere. **q.e.d.**

Exercise 2.10. *In this exercise we show that for distributions there is a one-to-one correspondence between solutions of the linear transport equation and initial values.*

1. *Show that for any distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$, the following distribution solves the equation $\partial_t \tilde{F} = 0$:*

   $\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ *with* $\tilde{F}(\phi) = F(\tilde{\phi})$ *and* $\tilde{\phi}(y,t) = \phi(y+bt,t)$ *for all* $(y,t) \in \mathbb{R}^n \times \mathbb{R}$.

2. *Show that the following formula defines a linear continuous map*

   $$I : C^\infty(\mathbb{R}^n \times \mathbb{R}) \to C_0^\infty(\mathbb{R}^n) \quad \textit{with} \quad I(\phi)(x) = \int_{\mathbb{R}} \phi(x,t)\,\mathrm{d}t.$$

3. *Let $\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ solve $\partial_t \tilde{F} = 0$. Show $\tilde{F}(\phi) = G(I(\phi))$ for some $G \in \mathcal{D}'(\mathbb{R}^n)$.*

4. *Show that for any mollifier* $(\lambda_\epsilon)_{\epsilon>0}$ *on* $\mathbb{R}$ *and any* $\phi \in C_0^\infty(\mathbb{R}^n)$ *the functions*

$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \textit{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$

belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ *and that* $\tilde{F}(\phi \times \lambda_\epsilon)$ *does not depend on* $\epsilon > 0$.

5. *Show that for any* $G \in \mathcal{D}'(\mathbb{R}^n)$ *the following* $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ *solves* $(\partial_t + b\nabla)F = 0$:

$$F : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}, \qquad \phi \mapsto G\left(\int_{\mathbb{R}} \mathsf{T}_{-tb}\phi(\cdot, t)\,\mathrm{d}t\right).$$

6. *Show that* $G \to F$ *is bijective onto* $\{F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \mid (\partial_t + b\nabla)F = 0\}$.

A short and lucid introduction into the theory of distributions is contained in the first chapter of the book of Lars Hörmander: "Linear Partial Differential Operators".

2.5 Regularity of Solutions

The regularity of a solution of a differential equation refers to the local properties of the corresponding functions. The most general functions we shall consider are distributions, which we say have the lowest regularity. They contain the measurable functions with the next highest regularity. The elements of L^p_{loc} describe ever smaller families of functions, whose regularity increase with $p \in [1, \infty]$. The next smallest class are Sobolev functions whose k-th order partial derivatives belong to L^p_{loc}. The regularity further increases for the functions in C^k. Finally we end with the smooth functions and the analytic functions with the highest regularity.

2.6 Boundary Value Problems

Our investigations of solutions of partial differential equations aims for a complete characterisations of all solutions. In general partial differential equations have an infinite dimensional space of solutions. A solution of an ordinary differential equations of m-th order is in many cases uniquely determined by fixing the values of the first m derivatives at some initial value of the parameter. For partial differential equations we search a similar characterisation. The solutions are functions on higher dimensional domains $\Omega \subset \mathbb{R}^n$. A natural condition is the specification of the values of the solution and some of its derivatives on the boundary of the domain. The search for solutions which obey this further specification are called boundary value problems. So an important objective in the investigation of partial differential equations is to find boundary value problems that have unique solutions. If we determine in addition all possible boundary values that have solutions, then the space of solutions is completely parametrised.