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Introduction to Partial Differential Equations
Exercise sheet 11

9th December, 2020

33. Scaling the heat kernel.

Find the heat kernel of

(a) $\mathbb{R}/c\mathbb{R}$ with $c > 0$, *(3 Point(s))*

(b) $[a, b]$ with $-\infty < a < b < \infty$. *(3 Point(s))*

34. Some like it hot.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - 7\partial_{xx} u = 0 & \text{for } x \in (0, \pi),\ t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\ u(x, 0) = 3\sin(2x) - 6\sin(5x) & \text{for } x \in (0, \pi). \end{cases}$$

(6 Point(s))

35. Out of the frying pan, into the fire.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - \partial_{xx} u = 0 & \text{for } x \in (0, \pi),\ t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\ u(x, 0) = x^2(\pi - x) & \text{for } x \in (0, \pi). \end{cases}$$

Further, show that your solution obeys $\displaystyle\int_0^\pi u(x, t)\, \mathrm{d}x = 8 \sum_{k \text{ odd}} \frac{1}{k^4} e^{-k^2 t}$. *(8 Bonus Point(s))*

36. Do nothing by halves ... again.

Consider the following heat equation with initial and boundary conditions, on the set $\Omega = (0, \infty)$:

$$\begin{cases} \dot{u} - \Delta u = 0 & \text{on } \Omega \times (0, \infty) \\ u(0, t) = 0 & \text{for } t \geq 0 \\ u(x, 0) = h(x) & \text{for } x \in \Omega, \end{cases}$$

where h is a bounded continuous function on $\overline{\Omega}$ with $h(0) = 0$. We are seeking a solution which is $C^2(\overline{\Omega} \times (0, \infty))$ and extends continuously to $\overline{\Omega} \times [0, \infty)$. Note, $C^k(A)$ for a non-open set A means that all derivatives up to an including order k (which are only defined on $\operatorname{int} A$) are continuous and also extend continuously to A.

(a) Suppose that we had such a solution $u(x, t)$. Extend both u and h to functions $\tilde{u} : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ and $\tilde{h} : \mathbb{R} \to \mathbb{R}$ that are odd in x, ie $\tilde{u}(x, t) = -\tilde{u}(-x, t)$ and $\tilde{h}(x) = -\tilde{h}(-x)$. What regularity must $\tilde{u}$ have? *(2 Point(s))*

(b) Suppose now that $\tilde{u}$ has sufficient regularity. Show that it solves the following Cauchy problem:

$$\begin{cases} \dot{v} - \Delta v = 0 & \text{on } \mathbb{R} \times (0, \infty) \\ v(x,0) = \tilde{h}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

(2 Point(s))

(c) Show that when u is a bounded solution of the PDE on the half-line, that $|u|$ is bounded by the supremum of $|h|$. Explain why it follows that a bounded solution is unique.

(2 Point(s))

(d) Show that

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(1 - e^{-\frac{xy}{t}}\right) \exp\left(-\frac{|x-y|^2}{4t}\right) h(y)\, \mathrm{d}y$$

is the unique bounded solution. *(3 Point(s))*

(e) Prove the estimate

$$|u(x,t)| \leq \frac{x}{\sqrt{4\pi}\, t^{3/2}} \int_0^\infty \exp\left(-\frac{|x-y|^2}{4t}\right) y\, |h(y)|\, \mathrm{d}y.$$

Hint: The exponential function is convex, so bounded by its tangent line.

(3 Point(s))

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.