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Introduction to Partial Differential Equations
Exercise sheet 9

25th November, 2020

27. Solutions of the homogeneous heat equation.

Let $u : \mathbb{R}^n \times (0\,\infty) \to \mathbb{R}$ be a solution of the homogeneous heat equation, i.e. $\dot{u} - \Delta u = 0$.

(a) Show for every $\lambda \in \mathbb{R}$ that $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ is a solution to the heat equation. *(2 Point(s))*

(b) Show that $v(x,t) := x \cdot \nabla u + 2t\,\dot{u}$ is also a solution. *(3 Point(s))*

(c) In the situation of $n = 1$, one spatial coordinate, let $v : \mathbb{R} \to \mathbb{R}$ be a given smooth function and $u(x,t) := v(t^{-1}x^2)$. Show that v is a solution of the differential equation

$$4z\,v''(z) + (2+z)\,v'(z) = 0 \quad \text{for } z > 0$$

exactly when u satisfies the heat equation. *(3 Point(s))*

28. An alternative description of the solutions of the heat equation.

Suppose that we are given an open region $\Omega \subset \mathbb{R}$ and an infinitely differentiable function $f : \Omega \to \mathbb{R}$. Suppose moreover that there is a constant $M > 0$ with

$$|(\Delta^k f)(x)| \le M^k,$$

for all $x \in \Omega$ and $k \ge 0$. Here Δ^k is the Laplace operator applied k-times. For example, $\Delta^2 f = \Delta(\Delta f)$. By convention, we set $\Delta^0 f = f$.

Show now that

$$u(x,t) := (e^{t\Delta})f(x) := \sum_{k=0}^{\infty} \frac{1}{k!}(\Delta^k f)(x)t^k$$

defines an infinitely differentiable function on $\Omega \times \mathbb{R}$ and further that it solves the initial value problem

$$\dot{u} - \Delta u = 0, \qquad u(x,0) = f(x).$$

(5 Point(s))

29. An alternative estimate for Corollary 3.4.

First let $\Omega' \subset \mathbb{R}^n \times \mathbb{R}$ be an open and connected region. A function $v : \Omega' \to \mathbb{R}$ is called a *sub-solution* of the heat equation if $\dot{v} - \Delta v \le 0$.

(a) *Mean value estimate for sub-solutions* Take any point $(x,t) in \Omega'$ and a small radius $r > 0$ so that $E(x,t,r) \in \Omega'$ (refer to Definition 4.6). Modify the proof the mean value property of the heat equation to show that

$$v(x,t) \le \frac{1}{4r^n} \int_{E(x,t,r)} v(y,s) \frac{|x-y|^2}{|t-s|^2}\, d^n y\, ds$$

holds for all sub-solutions. *(4 Point(s))*

Now let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and path connected region. We denote the parabolic cylinder of Ω by $\Omega_T := \Omega \times (0, T]$ as in Section 4.4. Suppose that $v : \Omega_T \to \mathbb{R}$ is a sub-solution that extends continuously to $\overline{\Omega_T}$.

(b) *Maximum principle for sub-solutions* Following on from (a), establish that if v takes the value $\sup_{\Omega_T} v$ on Ω_T, then it is constant. *(4 Point(s))*

(c) *A monotonicity property* For $j \in \{1, 2\}$ let $f_j : \Omega \times (0, T) \to \mathbb{R}$, $h_j : \Omega \to \mathbb{R}$, and $g_j : \partial\Omega \times [0, T]$ be smooth functions, and likewise let $u_j : \Omega \times (0, T)$ be smooth functions with continuous extensions to the boundary that satisfy

$$\begin{cases} \dot{u}_j - \Delta u_j = f_j & \text{on } \Omega \times (0, T) \\ u_j(x, 0) = h_j(x) & \text{on } \Omega \\ u_j = g_j & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

Suppose further that $f_1 \leq f_2$, $g_1 \leq g_2$, and $h_1 \leq h_2$. Show in this case that $u_1 \leq u_2$ as well. *(4 Point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.