Introduction to Partial Differential Equations
Exercise sheet 8

24. Do nothing by halves.

Let $H^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ be the upper half-space and $H^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ the dividing hyperplane. We call $R(x) = (x_1, \dots, x_{n-1}, -x_n)$ reflection in the plane H^0. Similarly $B^+ = B(0,1) \cap H^+$ and $B^0 = B(0,1) \cap H^0$.

(a) A reflection principle for harmonic functions. Let $u : \overline{B^+} \to \mathbb{R}^n$ be a harmonic function with $u|_{B^0} = 0$. Show that the function $v : \overline{B} \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_n \geq 0 \\ -u(R(x)) & \text{for } x_n < 0 \end{cases}$$

is harmonic. *(4 Point(s))*

(b) Green's function for the upper half-space. Show that Green's function for H^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R(x) - y).$$

(3 Point(s))

(c) Green's function for the half-ball. Compute the Green's function for B^+.
Hint: Make use of both the Green's function for the ball 3.20 and part (b).
(3 Point(s))

25. Teach a man to fish...

Using the Green's function of H^+ from the previous question, derive a formal integral representation for a solution of the Dirichlet problem

$$\Delta u = 0 \text{ in } H^+, \qquad u|_{H^0} = g.$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined.
(5 Point(s))

26. An alternative estimate for Corollary 3.4.

(a) Show the following estimate for all $x \neq 0$ and multiindices α:

$$|\partial^\alpha |x|^{-n}| \leq A(n, |\alpha|)\, |x|^{-n-|\alpha|},$$

where $A(n, |\alpha|)$ is a constant depending only on n and order $|\alpha|$. *(4 Point(s))*

(b) Hence give an alternative proof of Corollary 3.4 (you do not have to give a particular form for the constant).
Hint: Start from Poisson's representational formula. *(4 Point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.