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Introduction to Partial Differential Equations
Exercise sheet 7

11th November, 2020

21. Hackneyed Harnack.

Let $n \geq 3$, $r > 0$ and $B(0,r)$ the open ball in $\mathbb{R}^n$. Further, let $u : B(0,r) \to \mathbb{R}$ be a harmonic function with $u \geq 0$. Show that the following inequality holds for all $x \in B(0,r)$:

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0).$$

(5 Point(s))

22. Harmonic Polynomials.

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. A *real homogeneous polynomial of degree d* is a linear combination of monomials of the form $Q = x_1^{d_1} \dots x_n^{d_n}$ with $d_k \in \mathbb{N}_0$ and $d_1 + \cdots + d_n = d$. The vector space of real homogeneous polynomials of degree d is denoted $\mathcal{P}(d,n)$. We want to determine the dimension of the subspace of harmonic polynomials

$$\mathcal{H}(d,n) := \{P \in \mathcal{P}(d,n) \mid \Delta P = 0\}.$$

(a) Show using combinatorics that $\dim \mathcal{P}(d,n) = \binom{n+d-1}{d}$. *(2 Point(s))*

(b) Show that the Laplacian of a homogeneous polynomial of degree d is either zero or a homogeneous polynomial of degree $d-2$. *(2 Point(s))*

(c) Show that the linear map $\Delta : \mathcal{P}(d+2,n) \to \mathcal{P}(d,n)$ is surjective.
Hint: It suffices to show that every monomial is in the image of Δ. One may prove this by induction on n and d. *(5 Point(s))*

(d) What is the dimension of $\mathcal{H}(d,n)$? *(1 Point(s))*

23. Never judge a book by its cover.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1,2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(5 Point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.