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Introduction to Partial Differential Equations
Exercise sheet 6

4th November, 2020

17. Preparing the Mean Value Theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$, and $\partial B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$ for $r > 0$. Show that the function

$$F(r) := \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x)\, \mathrm{d}\sigma(x)$$

converges to $f(x_0)$ as $r \to 0$. *(4 Point(s))*

18. Harmonic Functions on $B(0,1) \subset \mathbb{R}^2$.

Let B be the open unit disc in $\mathbb{R}^2$.

(a) Let $u \in C^2(\overline{B})$ be a harmonic function which is given in polar coordinates $u = u(r, \varphi)$, with $0 \le r \le 1$ and $-\pi < \varphi \le \pi$. Show that in this coordinates

$$\int_{\partial B} \frac{\partial u}{\partial r}\, \mathrm{d}\sigma = 0.$$

(2 Point(s))

(b) "Guess" a solution $u \in C^2(\overline{B})$ for each of the following *Neumann Problems* or show that there is no solution

(i) $\Delta u = 0$ on B with $\frac{\partial u}{\partial r} = \sin\varphi$ on ∂B.

(ii) $\Delta u = 0$ on B with $\frac{\partial u}{\partial r} = \sin^2\varphi$ on ∂B.

(4 Point(s))

19. Inside Out.

(Optional) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be defined by

$$g(x) := \frac{1}{|x|^{n-2}} f\left(\frac{x}{|x|^2}\right).$$

Express Δg in terms of Δf, and thereby conclude that if f is a harmonic function then so too is g.

20. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A twice continuously differentiable function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if $-\Delta v \le 0$ in Ω.

(a) Let $v : \overline{\Omega} \to \mathbb{R}$ be subharmonic. Show that for all $x \in \Omega$ and $r > 0$ with $B(x, r) \subset \Omega$:

$$v(x) \le \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v(y)\, \mathrm{d}\sigma(y).$$

Hint: Show the function from Question 16 is monotonic. *(3 Point(s))*

(b) Conclude therefore the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 Point(s))*

(c) Now let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function.

- **(i)** Show that $\|\nabla u\|^2$ is subharmonic. *(2 Point(s))*
- **(ii)** Show that $f \circ u$ is subharmonic for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$. *(2 Point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.