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Introduction to Partial Differential Equations
Exercise sheet 5

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General Comment: Distributions are defined in Definition 2.6 as *continuous* linear maps. Recall that a linear map is continuous if and only if it is a bounded linear operator ($\Leftrightarrow \overline{F(B(0,1))}$ is compact). But the topology is a little complicated to describe in this situation, so please just use the condition given in the definition directly.

13. The Delta Quadrant.

(a) Show that
$$F : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_{\mathbb{R}} x^3 \cdot \phi''(x)\,\mathrm{d}x$$
is a distribution on $\mathbb{R}$, and define a function $f : \mathbb{R} \to \mathbb{R}$ with
$$F(\phi) = \int_{\mathbb{R}} f(x) \cdot \phi(x)\,\mathrm{d}x \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$
(4 Point(s))

(b) Show that the Dirac-Distribution
$$\delta : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \phi(0)$$
is indeed a distribution on $\mathbb{R}$ and prove that there does *not* exist a function $g : \mathbb{R} \to \mathbb{R}$ with
$$\delta(\phi) = \int_{\mathbb{R}} g(x) \cdot \phi(x)\,\mathrm{d}x \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$
(2+4 Point(s))

(c) Calculate the derivatives F' and δ' of the distributions in parts (a) und (b) respectively.
(2+2 Point(s))

14. An induced distribution.

Let $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ and $\psi \in C_0^\infty(\mathbb{R}^m)$. Define
$$G : C_0^\infty(\mathbb{R}^n) \to \mathbb{R}, \qquad \varphi \mapsto F(\varphi \times \psi).$$
Show that G is a Distribution on $C_0^\infty(\mathbb{R}^n)$, i.e. $G \in D'(\mathbb{R}^n)$.
(*Caution:* Don't forget to show, that G is well-defined.) *(5 Point(s))*

15. The Crucial Kernel.

When a the partial derivative of a function is zero, it is constant in that direction. In this question we investigate what it means when a distribution has a derivative that is zero. Let $F \in D'(\mathbb{R}^n \times \mathbb{R})$ and let (x,t) with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denote the elements in $\mathbb{R}^n \times \mathbb{R}$.
We want to show that: $\partial_t F = 0$ if and only if there is a distribution $G \in D'(\mathbb{R}^n)$ such that
$$F(\varphi) = G\Big(\int_{\mathbb{R}} \varphi(-,t)dt\Big).$$

From a certain point of view then, F does not depend on the t coordinate. In order to show the statement prove the following steps. First, define

$$\mathcal{I} : \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \to \mathcal{D}(\mathbb{R}^n),$$
$$\varphi \mapsto \left(x \mapsto \int_{-\infty}^{\infty} \varphi(x,t) \, \mathrm{d}t \right).$$

(a) (Optional) Show, that $\mathcal{I}$ is continuous and linear. *(3 Point(s))*

(b) Show that a function $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ belongs to the kernel of $\mathcal{I}$ if and only if it is the t-derivative of another such function. *(3 Point(s))*

(c) Show that for $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, $\partial_t F = 0$ if and only if $F \equiv 0$ on the kernel of $\mathcal{I}$. *(2 Point(s))*

(d) Finally show the statement by showing that $\partial_t F = 0$ if and only if there exists a $G \in \mathcal{D}'(\mathbb{R}^n)$ with $F(\varphi) = G(\mathcal{I}(\varphi))$. *(2 Point(s))*

16. You can now write "Transport-Distribution Expert" on your résumé.

In this exercise we show that there is a one-to-one correspondence between distributions solving the linear transport equation and distributions describing the corresponding initial values g.

(a) Show that for any distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$, the following distribution solves the equation $\partial_t \tilde{F} = 0$:

$$\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \text{ with } \tilde{F}(\phi) = F(\tilde{\phi}) \text{ and } \tilde{\phi}(y,t) = \phi(y - bt, t) \text{ for all } (y,t) \in \mathbb{R}^n \times \mathbb{R}.$$

(2 Point(s))

(b) Show that for any mollifier $(\lambda_\epsilon)_{\epsilon>0}$ on $\mathbb{R}$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$ the functions

$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \text{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$

belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. *(1 Point(s))*

(c) Recall $\mathcal{I}$ from the *The Crucial Kernel*. Let $\tilde{F} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solve the equation $\partial_t \tilde{F} = 0$. We have already proved that there exists a distribution $G \in \mathcal{D}(\mathbb{R}^n)$, such that $\tilde{F}(\phi) = G(\mathcal{I}(\phi))$. Argue therefore that $\tilde{F}(\phi \times \lambda_\epsilon)$ does not depend on $\epsilon > 0$. *(1 Point(s))*

(d) Show that for any $G \in \mathcal{D}(\mathbb{R}^n)$ the following $F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solves $(\partial_t + b\nabla)F = 0$:

$$F : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}, \qquad \phi \mapsto G\left(\int_{\mathbb{R}} \mathrm{T}_{-tb}\phi(\cdot, t) \, \mathrm{d}t \right),$$

where T_{-tb} is a translation operator. *(3 Point(s))*

(e) Show that $G \to F$ is bijective onto $\{F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \mid (\partial_t + b\nabla)F = 0\}$. *(3 Point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.