Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 4

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8. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C x \, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 point(s))*

(b) Consider the parametrisation of the circle $t \mapsto (\cos t, \sin t)$. Compute the integral in this parametrisation. *(2 point(s))*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1 - x^2})$. Compute the integral in this parametrisation. *(2 point(s))*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. [Hint. The easiest way is to use two parametrisations similar to part (b)] *(2 point(s))*

(e) Compute this integral using the divergence theorem. *(3 point(s))*

9. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the first Green formula

$$\int_\Omega v \triangle u \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \nabla u \cdot N \, d\sigma.$$

(3 points)

(b) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v \triangle u - u \triangle v) \, dx = \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N \, d\sigma.$$

(1 points)

10. The Black Spot.

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

(a) Show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

does not depend on the radius r, where N is the outward pointing normal. *(3 points)*

(b) What property of g explains this fact? In your proof, be careful to note that g is singular at $(0,0)$. *(3 points)*

(c) Prove for any compact region $\Omega \subset \mathbb{R}^2$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \, d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

(d) Comment on the flux of ∇g. *(1 points)*

11. Convoluted.

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y) \, dy.$$

(a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. *(2 Points)*

(b) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(2+3+4 Points)*

12. Go with the flow.

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity ρv, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold S is the integral

$$\int_S \rho v \cdot N \, d\sigma,$$

where N is the normal of S.

(a) Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

(b) How does this relate to the form of the conservation law derived in the lectures?

(c) For liquids a common property is *incompressibility*. For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that ρ is constant. However, slightly more general model says that ρ is not globally constant, but if we follow a point $x(t)$ along the velocity field v then $\rho(x(t),t)$ is constant. An example would be oil and water mixed together.
Use this description of incompressible flow to show that $\nabla \cdot v = 0$.

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are ‘Tiny Scanner’ and ‘Simple Scanner’.